Summary:

- We deduce additional properties of the Legendre polynomials introduced last week which enable us to use them to solve boundary-value problems, and give a few examples.
- We then introduce the *associated Legendre functions*, give some of their properties, and indicate how they are combined with the functions $\cos m\phi$ and $\sin m\phi$ which we saw last week to give a complete orthonormal set on a sphere.
- We then indicate how all of this is used to solve general (nonsymmetric) boundary-value problems for Laplace's equation on a sphere.

LEGENDRE POLYNOMIALS. Recall that the Legendre polynomials were defined last time as solutions to the differential equation

$$\left(1-x^2\right)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \ell(\ell+1)P = 0$$

as follows: seeking a power series solution $P = \sum_{n=0}^{\infty} a_n x^n$ of the above equation results in the recurrence relation for the coefficients

$$a_{n+2} = a_n \frac{n(n+1) - \ell(\ell+1)}{(n+2)(n+1)}.$$

If $\ell \in \mathbf{Z}$, $\ell \geq 0$, this says that all a_n for n of the same parity[1] as ℓ must eventually vanish. We then define the degree-ℓ *Legendre polynomial* as follows: If ℓ is even, let P_ℓ be the above power series solution with $a_1 = 0$ and a_0 chosen[2] so that $P_\ell(1) = 1$; if ℓ is odd, let P_ℓ be the above power series solution with $a_0 = 0$ and a_1 chosen so that $P_\ell(1) = 1$. Then we note the following properties:

- If ℓ is even, then P_ℓ is a sum of even powers of x and is hence an even function; if ℓ is odd, then P_ℓ is a sum of odd powers of x and is hence an odd function.
- From this, $P_\ell(-1) = (-1)^\ell$, and $P_\ell(0) = 0$ if ℓ is odd. ($P_e\ell(0)$ for *even* ℓ will be found below.)

Next we shall derive some results about the Legendre polynomials which are very useful in manipulating them. The proofs (except where noted) may be omitted without loss of continuity. Some of them are more advanced than the general level of this course.

The first result, while it may look strange at first,[3] is very useful (see the appendix for a similar result for the trigonometric functions and its use):

PROPOSITION. For $x \in [-1, 1]$ and $|h| < \frac{1}{4}$,

$$\left(1 - 2xh + h^2\right)^{-\frac{1}{2}} = \sum_{\ell=0}^{\infty} h^\ell P_\ell(x).$$

Proof. We note first that

$$1 - 2|x||h| + h^2 \leq 1 - 2xh + h^2 \leq 1 + 2|x||h| + h^2,$$

and since $0 \leq |x| \leq 1$, this shows that

$$(1 - |h|)^2 \leq 1 - 2xh + h^2 \leq (1 + |h|)^2,$$

so in particular $1 - 2xh + h^2 \geq \frac{9}{16} > 0$ and the function above is well-defined, and also $2xh - h^2 \leq \frac{7}{16} < \frac{1}{2}$ and $2xh - h^2 \geq 1 - (1 + |h|)^2 > -\frac{9}{16} > -1$; this last implies that for any fixed x in $[-1, 1]$ the above function can be expanded using the general binomial expansion theorem (exercise: prove this!)

$$(1 + h)^\alpha = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} h^n,$$

[1] By 'parity' we mean the property of a nonnegative integer according to which it is odd or even; in other words, two nonnegative integers are of the same parity if they are either both odd or both even.

[2] Note that a_0 is effectively an overall multiplicative constant in this case since we have only even-order terms.

[3] Some readers may note the relationship this result bears to the multipole expansion in electrostatics. This is not a coincidence!

where $|h| < 1$ and $(\alpha)_0 = 1$, $(\alpha)_{n+1} = (\alpha)_n \cdot (\alpha - n)$ (so that, for example, $(n)_n = n!$). In other words, for fixed $x \in [-1, 1]$ we may write

$$\left(1 - 2xh + h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2}\right)_n (-1)^n \left(2xh - h^2\right)^n;$$

by the standard theory of power series, the series on the left is uniformly and absolutely convergent also for $x \in [-1, 1]$, $|h| < \frac{1}{4}$, which means that we may reorder the terms as we wish. Doing so, we see easily that we get an expansion of the form

$$\left(1 - 2xh + h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} H_n(x) h^n,$$

where $H_n(x)$ is a polynomial in x. The fact that the above series converges uniformly in h for fixed x means that we may differentiate termwise with respect to h; since the original series may also be written in the form $\sum_{n=0}^{\infty} X(h) x^n$ for some polynomial X of h, and this series will also converge uniformly in x for fixed h,[4] we may also differentiate with respect to x termwise. We shall show that $H_n(x) = P_n(x)$, which will establish the result in the proposition, by showing that it satisfies Legendre's equation and has the correct normalisation. (By the foregoing, Legendre's equation has, up to normalisation, at most one polynomial solution for each ℓ.)

To do this, let $s(x, h) = \left(1 - 2xh + h^2\right)^{-\frac{1}{2}}$, and note that

$$\frac{\partial s}{\partial x} = h \left(1 - 2xh + h^2\right)^{-\frac{3}{2}} = hs^3, \qquad \frac{\partial^2 s}{\partial x^2} = 3h^2 s^5.$$

Similarly, note that

$$\frac{\partial}{\partial h} (hs) = s + h(x - h)s^3, \qquad \frac{\partial^2}{\partial h^2} (hs) = (x - h)s^3 + 3h(x - h)^2 s^5 + (x - 2h)s^3,$$

whence we see that

$$\begin{aligned}(1 - x^2) \frac{\partial^2 s}{\partial x^2} - 2x \frac{\partial s}{\partial x} &= (1 - x^2) \cdot 3h^2 s^5 - 2xhs^3 = s^5 \left(3h^2(1 - x^2) - 2xh(1 - 2xh + h^2)\right) \\ &= s^5 \left(3h^2 + h^2 x^2 - 2xh - 2xh^3\right)\end{aligned}$$

,

while

$$\begin{aligned}\frac{\partial^2}{\partial h^2} (hs) &= s^5 \left(3h(x - h)^2 + (2x - 3h)(1 - 2xh + h^2)\right) \\ &= s^5 \left(3h(h^2 - 2xh + x^2) + 2x(1 - 2xh + h^2) - 3h(1 - 2xh + h^2)\right) \\ &= s^5 \left(3hx^2 + 2x - 4x^2 h + 2h^2 x - 3h\right) \\ &= s^5 \left(-hx^2 + 2x + 2h^2 x - 3h\right),\end{aligned}$$

whence we see that

$$-h \frac{\partial^2}{\partial h^2} (hs) = (1 - x^2) \frac{\partial^2 s}{\partial x^2} - 2x \frac{\partial s}{\partial x}.$$

But by the power series expansion, we have

$$-h \frac{\partial^2}{\partial h^2} (hs) = -h \sum_{n=0}^{\infty} H_n(x) n(n + 1) h^{n-1} = -\sum_{n=0}^{\infty} n(n + 1) H_n(x) h^n,$$

[4] Note that we can allow x to lie in an open interval slightly larger than $[-1, 1]$ without changing the foregoing arguments, since $|h| < \frac{1}{4}$ kept us well inside the radius of convergence of the binomial expansion theorem; this justifies the uniform convergence just claimed.

whence we see that H_n is a polynomial satisfying Legendre's equation, as claimed. To check its normalisation, set $x = 1$ in s; then we have

$$\sum_{n=0}^{\infty} H_n(1)h^n = \left(1 - 2h + h^2\right)^{-\frac{1}{2}} = \frac{1}{1-h},$$

which implies that $H_n(1) = 1$ for all n. Thus we must have $H_n(x) = P_n(x)$ for $x \in [-1, 1]$, as claimed.QED.

The function s in this result is called the *generating function* for the Legendre polynomials. From this result the five identities given in class can be easily derived, as follows.

PROPOSITION. The Legendre polynomials satisfy the following identities (where $n \in \mathbf{Z}$, $n \geq 0$, and we set $P_{-1} = 0$):

1. $(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0$.
2. $P'_{n+1} - 2xP'_n + P'_{n-1} = P_n$.
3. $xP'_n - P'_{n-1} = nP_n$.
4. $P'_{n+1} - P'_{n-1} = (2n+1)P_n$.
5. $(1-x^2)P'_n = nP_{n-1} - nxP_n$.

Proof. We note that (letting, as above, $s = \left(1 - 2xh + h^2\right)^{-\frac{1}{2}}$)

$$\begin{aligned}\frac{\partial s}{\partial h} &= (x-h)s^3 = (x-h)(1-2xh+h^2)^{-1}s \\ &= \sum_{n=0}^{\infty} nP_n(x)h^{n-1},\end{aligned}$$

whence we see that

$$\begin{aligned}(x-h)\sum_{n=0}^{\infty} P_n(x)h^n &= \sum_{n=0}^{\infty}\left(xP_n(x) - P_{n-1}\right)h^n = (x-h)s \\ &= (1-2xh+h^2)\sum_{n=0}^{\infty} nP_n(x)h^{n-1} \\ &= \sum_{n=0}^{\infty}\left((n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)\right)h^n\end{aligned}$$

(recall our convention that $P_{-1} = 0$; this was used twice in the above calculation) from which we obtain

$$\begin{gathered}xP_n - P_{n-1} = (n+1)P_{n+1} - 2xnP_n + (n-1)P_{n-1} \\ (n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0,\end{gathered}$$

proving the first identity. Similarly, differentiating s with respect to x gives

$$\begin{aligned}\frac{\partial s}{\partial x} &= hs^3 = h(1-2xh+h^2)^{-1}s \\ &= \sum_{n=0}^{\infty} P'_n h^n,\end{aligned}$$

whence we have (noting that this last series has no $n = 0$ term since $P'_0 = 0$, so that we may advance its index by 1)

$$\begin{aligned}hs = \sum_{n=0}^{\infty} P_n h^{n+1} &= (1-2xh+h^2)\sum_{n=0}^{\infty} P'_{n+1}h^{n+1} \\ &= \sum_{n=0}^{\infty}\left(P'_{n+1} - 2xP'_n + P'_{n-1}\right)h^{n+1},\end{aligned}$$

from which the second identity easily follows. Now multiply the second identity by $n+1$ and subtract it from the derivative of the first identity to obtain

$$-(n+1)P_n = (n+1)P'_{n+1} - (2n+1)xP'_n - (2n+1)P_n + nP'_{n-1} - \left((n+1)P'_{n+1} - (2n+2)xP'_n + (n+1)P'_{n-1}\right)$$
$$xP'_n - (2n+1)P_n - P'_{n-1} = -(n+1)P_n$$
$$xP'_n - P'_{n-1} = nP_n,$$

which is the third identity. Adding twice the third identity to the second identity gives the fourth identity. Finally, to obtain the fifth identity, note that adding the second and fourth identities gives $2P'_{n+1} - 2xP'_n = 2(n+1)P_n$, which, upon dividing by 2 and replacing $n+1$ by n, gives

$$P'_n = xP'_{n-1} + nP_{n-1},$$

and thus, from the third identity,

$$(1-x^2)P'_n = P'_n - x\left(P'_{n-1} + nP_n\right) = -nxP_n + nP_{n-1},$$

which is exactly the fifth identity. QED.

We also have *the Rodrigues formula:*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

We shall not prove this at the moment (it can be proved by expanding out $(x^2-1)^n$ using the standard binomial expansion theorem, differentiating term-by-term, showing that the resulting coefficients of the powers of x satisfy the same recursion relation as the coefficients for the Legendre polynomials, and then checking the normalisation at 1).

From the Rodrigues formula we may deduce the orthogonality property of the Legendre polynomials:

PROPOSITION. We have

$$\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\,dx = \begin{cases} 0, & \ell \neq \ell' \\ \frac{2}{2\ell+1}, & \ell = \ell' \end{cases}.$$

Proof. Suppose that $\ell \geq \ell'$; then we have, applying the Rodrigues formula and integrating by parts (it can be shewn that the boundary terms all vanish)

$$\begin{aligned}\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\,dx &= \frac{1}{2^\ell \ell!}\frac{1}{2^{\ell'}\ell'!}\int_{-1}^{1}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\frac{d^{\ell'}}{dx^{\ell'}}(x^2-1)^{\ell'}\,dx \\ &= -\frac{1}{2^\ell \ell!}\frac{1}{2^{\ell'}\ell'!}\int_{-1}^{1}\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\frac{d^{\ell'+1}}{dx^{\ell'+1}}(x^2-1)^{\ell'}\,dx \\ &= (-1)^\ell\frac{1}{2^\ell \ell!}\frac{1}{2^{\ell'}\ell'!}\int_{-1}^{1}(x^2-1)^\ell\frac{d^{\ell'+\ell}}{dx^{\ell'+\ell}}(x^2-1)^{\ell'}\,dx.\end{aligned}$$

Now if $\ell' \neq \ell$, then since $\ell \geq \ell'$ we must have $\ell > \ell'$; thus $\ell' + ell > 2\ell'$, but since $(x^2-1)^{\ell'}$ is a polynomial of degree $2\ell'$ this implies that $\frac{d^{\ell'+\ell}}{dx^{\ell'+\ell}}(x^2-1)^{\ell'} = 0$ identically and the above integral must be zero, as claimed. If $\ell' = \ell$, then the foregoing derivative is simply the constant $(2\ell)!$; finishing the proof requires integrating $\int_{-1}^{1}(x^2-1)^\ell\,dx$, which requires the use of a trigonometric reduction formula and which we pass over for the time being. QED.

The foregoing shows that the set $\{P_\ell | \ell \in \mathbf{Z},\quad \ell \geq 0\}$ is an orthogonal set on $[-1,1]$; it can be shewn that it is complete. Thus any (suitably nice; e.g., piecewise continuous) function f on the interval $[-1,1]$ can be expanded uniquely in a series

$$f(x) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(x),$$

where by our work with general orthogonal complete sets at the start of the course we have the usual formula

$$a_\ell = \frac{(f, P_\ell)}{(P_\ell, P_\ell)} = \frac{2\ell+1}{2}(f, P_\ell).$$

We would now like to know how this helps us solve boundary-value problems. Suppose that we are asked to solve Laplace's equation in a spherical shell $a < r < b$, with azimuthally symmetric boundary data given on the inner and outer spheres $r = a$ and $r = b$. On the interior region, the solution will be given by the general expression

$$u(r, \theta) = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \left(a_\ell r^\ell + b_\ell r^{-\ell-1}\right).$$

It is instructive to compare this to the general expression

$$u(x, y) = \sum_{n=1}^{\infty} \sin n\pi x \left(a_n \sinh n\pi y + b_n \cosh n\pi y\right)$$

which we obtained for the solution to Laplace's equation in the unit square with homogeneous Dirichlet data on the two vertical boundaries (i.e., $u(0, y) = u(1, y) = 0$). As we did on the unit square, we now apply the boundary conditions on the inner and outer spheres to fix the coefficients a_ℓ and b_ℓ. More concretely, suppose that we are given $u(a, \theta) = f(\theta)$, $u(b, \theta) = g(\theta)$; then we may write

$$f(\theta) = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \left(a_\ell a^\ell + b_\ell a^{-\ell-1}\right),$$
$$g(\theta) = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \left(a_\ell b^\ell + b_\ell b^{-\ell-1}\right).$$

We would like to use the fact that P_ℓ is complete on $[-1, 1]$ in order to evaluate the coefficients of the above series. This requires a little bit of work though since the series above are in terms of $P_\ell(\cos\theta)$ while the functions f and g are given in terms of θ itself. Note that since $\theta \in [0, \pi]$, we have $\cos\theta \in [-1, 1]$, so that (letting $x = \cos\theta$ as before) we have $\cos^{-1} x = \theta \in [0, \pi]$ for $x \in [-1, 1]$; this implies that we may expand $f(\theta) = f(\cos^{-1} x)$ and $g(\theta) = g(\cos^{-1} x)$ in series of $P_\ell(x) = P_\ell(\cos\theta)$ using the above formula. To do this, we calculate

$$\begin{aligned}(f \circ \cos^{-1}, P_\ell) &= \int_{-1}^{1} f(\cos^{-1}(x)) P_\ell(x)\, dx \\ &= \int_0^\pi f(\theta) P_\ell(\cos\theta) \sin\theta\, d\theta,\end{aligned}$$

where we have changed variables in the integral in the last step. This shows that in an expansion of the form

$$f(\theta) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(\cos\theta),$$

we have

$$c_\ell = \frac{2\ell+1}{2} \int_0^\pi f(\theta) P_\ell(\cos\theta) \sin\theta\, d\theta.$$

Another way of saying this is that the functions $P_\ell(\cos\theta)$ are orthogonal on the interval $[0, \pi]$ with respect to the inner product

$$(f, g) = \int_0^\pi f(\theta) \overline{g(\theta)} \sin\theta\, d\theta.$$

Thus we obtain finally the system of equations

$$\begin{aligned}
a_\ell a^\ell + b_\ell a^{-\ell-1} &= \frac{2\ell+1}{2}\int_0^\pi f(\theta)P_\ell(\cos\theta)\sin\theta\,d\theta,\\
a_\ell b^\ell + b_\ell b^{-\ell-1} &= \frac{2\ell+1}{2}\int_0^\pi g(\theta)P_\ell(\cos\theta)\sin\theta\,d\theta,
\end{aligned}$$

which when solved will give us a_ℓ and b_ℓ for all ℓ, from which the solution to the desired problem follows.

Other problems (for example, when the boundary data involves the derivatives u_r) can be solved in a similar way.

It is worthwhile pausing for a moment to note a general pattern here which will come up again in the future: a given polynomial $P_\ell(\cos\theta)$ on the boundary will give a solution varying like $P_\ell(\cos\theta)r^\ell$ or $P_\ell(\cos\theta)r^{-\ell-1}$ in the interior, and the 'amount' of a certain Legendre polynomial P_ℓ in given boundary data determines exactly the 'amount' of that polynomial in the final solution. In other words, if we think of the coefficients in the expansions of f and g above in terms of $P_\ell(\cos\theta)$ as being knobs we can turn, then it is as if each knob corresponds to a particular type of behaviour of the full solution, and fixing boundary data is equivalent to fixing the position of each knob. The specific way in which the knobs control the solution is determined by solving the equations above.

FULL SOLUTIONS TO LAPLACE'S EQUATION. Let us now consider the problem of finding solutions to Laplace's equation in the absence of azimuthal symmetry. Recalling our results from applying separation of variables to Laplace's equation in spherical symmetry, we see that in this case Legendre's equation is replaced by the equation (still writing $x = \cos\theta$)

$$\left(1-x^2\right)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P = 0,$$

while if $P_{\ell m}$ is a solution to this equation, then the corresponding general separated solution to Laplace's equation is given by

$$P_{\ell m}(\cos\theta)\left(a_{\ell m}r^\ell + b_{\ell m}r^{-\ell-1}\right)\left(c_{\ell m}\cos m\phi + d_{\ell m}\sin m\phi\right).$$

Solutions to this equation may be found by the following trick. We differentiate Legendre's equation m times:

$$\begin{aligned}
\left(1-x^2\right)P''' - 4xP'' + \left(\ell(\ell+1) - 2\right)P' &= 0\\
\left(1-x^2\right)P^{(4)} - 6xP''' + \left(\ell(\ell+1) - 2 - 4\right)P'' &= 0\\
\left(1-x^2\right)P^{(5)} - 8xP^{(4)} + \left(\ell(\ell+1) - 2 - 4 - 6\right)P'' &= 0\\
&\vdots\\
\left(1-x^2\right)P^{(m+2)} - 2(m+1)xP^{(m+1)} + \left(\ell(\ell+1) - m(m+1)\right)P^{(m)} &= 0,
\end{aligned}$$

since $2+4+6+8+\cdots+2m = 2\frac{m(m+1)}{2} = m(m+1)$. Thus

$$\begin{aligned}
&\frac{d}{dx}\left(\left(1-x^2\right)\frac{d}{dx}\left(\left(1-x^2\right)^{\frac{m}{2}}P^{(m)}\right)\right)\\
&\quad= \frac{d}{dx}\left[\left(1-x^2\right)^{\frac{m}{2}+1}P^{(m+1)} - mx\left(1-x^2\right)^{\frac{m}{2}}P^{(m)}\right]\\
&\quad= \left(1-x^2\right)^{\frac{m}{2}+1}P^{(m+2)} - m\left(1-x^2\right)^{\frac{1}{m}}P^{(m)} + m^2x^2\left(1-x^2\right)^{\frac{m}{2}-1}P^{(m)}\\
&\qquad\qquad - (m+m+2)\left(1-x^2\right)^{\frac{m}{2}}P^{(m+1)}\\
&\quad= \left(1-x^2\right)^{\frac{m}{2}}\left[\left(1-x^2\right)P^{(m+2)} - 2(m+1)xP^{(m+1)} - \left(m - \frac{m^2x^2}{1-x^2}\right)P^{(m)}\right]\\
&\quad= \left(1-x^2\right)^{\frac{m}{2}}\left[m(m+1) - \ell(\ell+1) - m + \frac{m^2x^2}{1-x^2}\right]P^{(m)} = \left[\frac{m^2}{1-x^2} - \ell(\ell+1)\right]\left(1-x^2\right)^{\frac{m}{2}}P^{(m)};
\end{aligned}$$

comparison with the equation we are trying to solve shows that it has the solution

$$P_{\ell,m}(x) = (1-x^2)^{\frac{m}{2}} P_\ell^{(m)}(x).$$

We call these the *associated Legendre functions*. The corresponding general separated solution to Laplace's equation is (as given above)

$$P_{\ell m}(\cos\theta)\left(a_{\ell m} r^\ell + b_{\ell m} r^{-\ell-1}\right)\left(c_{\ell m}\cos m\phi + d_{\ell m}\sin m\phi\right).$$

It is worthwhile to pause for a moment to consider what these functions look like, and what the corresponding solutions to Laplace's equation look like. First, since P_ℓ is a polynomial of degree ℓ, we see that $P_\ell^{(m)}$ will vanish if $m > \ell$; thus we require that $m \le \ell$.[5] Further, $P_{\ell,0} = P_\ell$ is just the ordinary Legendre polynomial. The first few additional associated Legendre functions may be calculated as follows. It is instructive to evaluate them at $\cos\theta$ (though we must bear carefully in mind that the derivatives in $P_\ell^{(m)}$ are with respect to x, not θ!), noting that since $\theta \in [0,\pi]$, $(1-\cos^2\theta)^{\frac{1}{2}} = (\sin^2\theta)^{\frac{1}{2}} = |\sin\theta| = \sin\theta$.

$$\begin{aligned}
P_{1,1}(\cos\theta) &= \sin\theta \\
P_{2,1}(\cos\theta) &= 3\sin\theta\cos\theta \\
P_{2,2}(\cos\theta) &= 3\sin^2\theta.
\end{aligned}$$

Thus we see that increasing m by one essentially trades a $\cos\theta$ for a $\sin\theta$. Now the corresponding solutions to Laplace's equation on a ball (meaning that we disregard the $r^{-\ell-1}$ solutions) are of the form

$$\begin{aligned}
P_{1,1}(\cos\theta) r\cos\phi &= \sin\theta r\cos\phi = r\sin\theta\cos\phi = x \\
P_{2,1}(\cos\theta) r^2\cos\phi &= 3r^2\sin\theta\cos\theta\cos\phi = 3xz \\
P_{2,2}(\cos\theta) r^2\cos 2\phi &= 3r^2\sin^2\theta\left(\cos^2\phi - \sin^2\phi\right) = 3\left(x^2-y^2\right),
\end{aligned}$$

with similar expressions obtaining if the $\sin\phi$ and $\sin 2\phi$ solutions are used instead. The polynomials of the form $P_{\ell,m} r^\ell \cos m\phi$ and $P_{\ell,m} r^\ell \sin m\phi$,[6] which are all solutions to Laplace's equation, are called *harmonic polynomials*. As we shall see shortly, the set of all products of the form $P_{\ell,m}\cos m\phi$, $P_{\ell,m}\sin m\phi$ forms a complete orthogonal set over the sphere; since on a sphere r is a constant, this implies that given any polynomial on $\mathbf{R}^n$ and a sphere of radius $r = a$ centred at the origin, there will be a harmonic polynomial which agrees with the given polynomial on the sphere. This harmonic polynomial will then be the solution to Laplace's equation with boundary data equal to the given polynomial. This is an interesting branch of mathematics but we shall not explore it in detail here.

We would now like to see how we can use the associated Legendre functions to solve boundary-value problems for Laplace's equation on a sphere. First, we note that the $P_{\ell,m}$, for constant m, form an orthogonal set; in particular, we have the following result.

PROPOSITION. Let $m \in \mathbf{Z}$, $m \ge 0$, $\ell_1, \ell_2 \in \mathbf{Z}$, $\ell_1, \ell_2 \ge m$, $\ell_1 \ne \ell_2$. Then

$$\int_{-1}^{1} P_{\ell_1,m} P_{\ell_2,m}\, dx = 0.$$

[NOTE. By the same logic as we used above for the Legendre polynomials, in terms of θ the above orthogonality result becomes

$$\int_{-1}^{1} P_{\ell_1,m}(\cos\theta) P_{\ell_2,m}(\cos\theta)\sin\theta\, dx = 0.]$$

[5]This requirement might be familiar to those of you who have studied the theory of angular momentum in quantum mechanics. What we are building here are parts of the angular momentum eigenfunctions in the position representation.

[6]These may be handled simultaneously by using the complex form $P_{\ell,m} r^\ell e^{im\phi}$, in the which case the general form becomes more transparent.

Proof. We use a general method (which can also be applied to the Legendre polynomials themselves), by showing that the differential operator

$$f \mapsto Lf := \frac{d}{dx}\left(\left(1-x^2\right)\frac{df}{dx}\right)$$

is what is called *self-adjoint*, i.e., that if $f, g \in C^2$, then $(Lf, g) = (f, Lg)$. This may be shewn as follows:

$$\begin{aligned}(Lf, g) &= \int_{-1}^{1} [Lf](x) \cdot \overline{g(x)}\, dx = \int_{-1}^{1} \frac{d}{dx}\left(\left(1-x^2\right)\frac{df}{dx}\right)\overline{g(x)}\, dx \\ &= \left(1-x^2\right)\frac{df}{dx}\overline{g(x)}\Big|_{-1}^{1} - \int_{-1}^{1}\left(1-x^2\right)\frac{df}{dx}\frac{\overline{dg}}{dx}\, dx \\ &= -\, f(x)\left(1-x^2\right)\frac{\overline{dg}}{dx}\Big|_{-1}^{1} + \int_{-1}^{1} f(x)\overline{\frac{d}{dx}\left(\left(1-x^2\right)\frac{dg}{dx}\right)}\, dx \\ &= (f, Lg),\end{aligned}$$

as claimed. Now let $m \in \mathbf{Z}$, $m \geq 0$. It is clear that the operator $f \mapsto Mf := \frac{m^2}{1-x^2}f$ also satisfies this property, and hence so does the difference $L' = L - M$. Now let $P_{\ell_1,m}$ and $P_{\ell_2,m}$ be as in the statement of the proposition. Then we have $L'P_{\ell_1,m} = -\ell_1(\ell_1+1)P_{\ell_1,m}$ and $L'P_{\ell_2,m} = -\ell_2(\ell_2+1)P_{\ell_2,m}$; thus

$$\begin{aligned}(L'P_{\ell_1,m}, P_{\ell_2,m}) &= -\ell_1(\ell_1+1)(P_{\ell_1,m}, P_{\ell_2,m}) \\ &= (P_{\ell_1,m}, L'P_{\ell_2,m}) = -\ell_2(\ell_2+1)(P_{\ell_1,m}, P_{\ell_2,m}),\end{aligned}$$

whence $(P_{\ell_1,m}, P_{\ell_2,m}) = 0$ since $\ell_1 \neq \ell_2$. QED.

It may also be shewn that (see (4.2.25) in the textbook)

$$(P_{\ell,m}, P_{\ell,m}) = \frac{(\ell+m)!}{(\ell-m)!}\frac{2}{2\ell+1}. \tag{1}$$

We now recall that the set

$$\{1, \cos m\phi, \sin m\phi | m \in \mathbf{Z}, m \geq 1\}$$

is a complete orthogonal set on $[0, 2\pi]$ (while we may not have explicitly shewn this earlier, it follows readily from what we have done). We claim that this, together with the above proposition, implies that

$$\{P_\ell(\cos\theta)\} \cup \{P_{\ell,m}(\cos\theta)\cos m\phi, P_{\ell,m}(\cos\theta)\sin m\phi | m \in \mathbf{Z}, m \geq 1\}$$

is a complete orthogonal set on the sphere (i.e., on the set $[0,\pi] \times [0, 2\pi]$ with respect to the inner product

$$(f(\theta,\phi), g(\theta,\phi)) = \int_0^\pi \int_0^{2\pi} f(\theta, phi)\overline{g(\theta,\phi)}\sin\theta\, d\phi\, d\theta.$$

The orthogonality is simple to shew. We note first of all that if $f(\theta,\phi) = f_1(\theta)f_2(\phi)$ and $g(\theta, phi) = g_1(\theta)g_2(\phi)$, then the above inner product decomposes as follows:

$$(f, g) = \int_0^\pi f_1(\theta)\overline{g_1(\theta)}\sin\theta\, d\theta \int_0^{2\pi} f_2(\phi)\overline{g_2(\phi)}\, d\phi;$$

in other words, it is simply the product of the inner products we have been using for functions of θ and ϕ.[7] Thus

$$(P_{\ell,m}\cos m\phi, P_{\ell',m'}\cos m'\phi) = (P_{\ell,m}, P_{\ell',m'})(\cos m\phi, \cos m'\phi) = 0$$

[7]This is related to the fact that the space of functions on the sphere can be viewed as the tensor product (in an appropriate sense: one needs to somehow close off the algebraic tensor product in an appropriate topology, such as that given by the product inner product indicated above) of the spaces of functions on $[0, 2\pi]$ and $[0, \pi]$.

if $m \neq m'$, while if $m = m'$ it will be zero unless $\ell = \ell'$, in the which case it will be given by the normalisation formula (1) above. An analogous result clearly holds if both of the cosine terms are replaced by sine, while if one is replaced by sine then all of the inner products will be zero regardless of m. This shews that the set above is an orthogonal set, as desired.

To see that it is complete (assuming that $\{P_{\ell,m}\}$ and the sine/cosine basis are), we may proceed as follows. Let $f(\theta, \phi)$ be any suitable (e.g., piecewise continuous) function on the sphere. Then for each fixed θ we may expand it in a series of sines and cosines, the coefficients of which will however depend on θ; in other words, we may write[8]

$$f(\theta, \phi) = \sum_{m=0}^{\infty} c_m(\theta) \cos m\phi + d_m(\theta) \sin m\phi;$$

now since the $P_{\ell,m}$, for each m, form a complete set on $[0, \pi]$, we may further expand each of the coefficients c_m, d_m, obtaining

$$\begin{aligned} c_m(\theta) &= \sum_{\ell=m}^{\infty} c_{\ell,m} P_{\ell,m}(\cos\theta), \\ d_m(\theta) &= \sum_{\ell=m}^{\infty} d_{\ell,m} P_{\ell,m}(\cos\theta). \end{aligned}$$

Thus we have finally

$$f(\theta, \phi) = \sum_{m=0}^{\infty} \sum_{\ell=m}^{\infty} P_{\ell,m}(\cos\theta) \left(c_{\ell,m} \cos m\phi + d_{\ell,m} \sin m\phi \right),$$

or, assuming that the series is sufficiently well-behaved that we may rearrange the order of the terms,

$$f(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} P_{\ell,m}(\cos\theta) \left(c_{\ell,m} \cos m\phi + d_{\ell,m} \sin m\phi \right),$$

where by the orthogonality result above

$$\begin{aligned} c_{\ell,m} &= \frac{(f, P_{\ell,m}(\cos\theta)\cos m\phi)}{(P_{\ell,m}(\cos\theta)\cos m\phi, P_{\ell,m}(\cos\theta)\cos m\phi)} \\ &= \frac{2\ell+1}{2\pi} \frac{(\ell-m)!}{(\ell+m)!} (f, P_{\ell,m}(\cos\theta)\cos m\phi), \end{aligned}$$

with an analogous formula for $d_{\ell,m}$.

Our procedure for solving general problems involving Laplace's equation on spherical shells is now clear: we start out with the general series representation

$$\sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} P_{\ell m}(\cos\theta) \left(a_{\ell m} r^{\ell} + b_{\ell m} r^{-\ell-1} \right) \left(c_{\ell m} \cos m\phi + d_{\ell m} \sin m\phi \right),$$

and then apply the boundary conditions, using formulas like the above for $c_{\ell,m}$ to determine equations for the relevant coefficients (exactly as we did when solving Laplace's equation on a square), to determine all of the coefficients in the expansion. We then substitute back in to obtain the desired solution.

As we are well aware by now, this process in general produces very long expressions; also, the integrals arising for general boundary data can be very difficult to evaluate. Some simple examples can be done by exploiting the idea of harmonic polynomial mentioned earlier; we give one such example.

[8]Here and below, to avoid having to pull out the $m = 0$ term explicitly, we shall make the definition that d_0, etc., are all zero.

EXAMPLE. Solve Laplace's equation $\nabla^2 u = 0$ on the unit ball with boundary data $u|_{r=1} = \cos\theta \sin\theta \sin\phi + \sin^2\theta \sin\phi \cos\phi$.

First of all, we note that since we are solving on the interior of a sphere, our solution must be continuous at the origin, so the terms $r^{-\ell-1}$ cannot appear and we must have $b_{\ell,m} = 0$ for all ℓ, m; we may thus absorb the coefficients $a_{\ell,m}$ into the $c_{\ell,m}$ and $d_{\ell,m}$. We could proceed by rewriting the above boundary data as a linear combination of products of associated Legendre functions with functions $\cos m\phi$, $\sin m\phi$; this would give

$$\frac{1}{3}P_{2,1}\sin\phi + \frac{1}{6}P_{2,2}\sin 2\phi,$$

from which we would obtain the solution

$$\frac{1}{3}P_{2,1}r^2\sin\phi + \frac{1}{6}P_{2,2}r^2\sin 2\phi.$$

Alternatively, we may note that on $r = 1$ the above expression is equal to the polynomial

$$zy + xy,$$

which satisfies Laplace's equation everywhere through space; thus our solution is simply $u = zy + xy$, as can be verified from the first expression given above.