Summary:

- When solving problems with boundary data specified on circles, cylinders, or spheres, it is useful to work in coordinate systems adapted to the boundary surfaces at hand.
- The gradient in cylindrical coordinates is given by

$$\nabla f = \frac{\partial f}{\partial \rho}\boldsymbol{\rho} + \frac{1}{r}\frac{\partial f}{\partial \phi}\boldsymbol{\phi} + \frac{\partial f}{\partial z}\mathbf{k},$$

  and in spherical coordinates by

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{\phi}.$$

- The divergence in cylindrical coordinates of a vector field $\mathbf{F} = F_\rho \boldsymbol{\rho} + F_\phi \boldsymbol{\phi} + F_z \mathbf{k}$ is given by

$$\frac{\partial F_\rho}{\partial \rho} + \frac{1}{\rho}F_\rho + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z},$$

  and the divergence in spherical coordinates of a vector field $\mathbf{F} = F_r\mathbf{r} + F_\theta\boldsymbol{\theta} + F_\phi\boldsymbol{\phi}$ is given by

$$\frac{\partial F_r}{\partial r} + \frac{2}{r}F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{1}{r}\cot\theta F_\theta + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}.$$

- In cylindrical coordinates, Laplace's equation becomes

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

  and in spherical coordinates,

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0.$$

- When we separate variables in Laplace's equation in spherical coordinates, we get solutions $u = R\Theta\Phi$, where R, Θ, and Φ are of the following form:

$$R = ar^\ell + br^{-(\ell+1)}, \qquad \Theta = P_\ell^m(\cos\theta), \qquad \Phi = c\cos m\theta + d\sin m\theta,$$

  where ℓ and m are nonnegative integers and P_ℓ^m is a *Legendre function*. The simplest case is when $m = 0$, in the which case we write $\Theta = P_\ell(\cos\theta)$, where P_ℓ is the *Legendre polynomial of degree* ℓ.

MOTIVATION. We have by now seen a few examples of the use of the separation-of-variables technique to solve Laplace's equation on a square. Exactly similar methods would work to solve it on a rectangle, and in three (or even higher) dimensions we could solve it on a cube with exactly analogous techniques. Suppose however that our boundary data were given on a circle, or a sphere – this would be a very different matter. Thinking back to our general series solution to Laplace's equation on the unit square,

$$u(x,y) = \sum_{n=1}^\infty \sin n\pi x\left(a_n \sinh n\pi y + b_n \cosh n\pi y\right),$$

if we were given boundary data on a circle, we would need to satisfy a requirement of the form

$$u(x, \sqrt{1-x^2}) = f(x) = \sum_{n=1}^\infty \sin n\pi x\left(a_n \sinh n\pi\sqrt{1-x^2} + b_n \cosh n\pi\sqrt{1-x^2}\right),$$

and now not only does it look hopeless to try to integrate this series against $\sin m\pi x$, it seems pretty clear that that is not even the right thing to try since now y depends on x rather than being constant, and it is not at all clear that integrating against $\sin m\pi x$ will allow us to deduce the expansion coefficients a_n and b_n. Thus it seems that in cases like this something else is required. It turns out that the correct way forwards is to do a change of variables and work in polar, cylindrical, or spherical coordinates. This is analogous to how we change integrals to integrate over circular or spherical regions in multivariable calculus.

NOTE. The derivations of the expressions for the gradient and divergence below are rather technical. Since in this class we only really need the end results of these derivations, i.e., the expressions for the Laplacian in spherical and cylindrical coordinates, the derivations themselves are of secondary importance and may be skipped without essential loss of continuity. They are given here for the sake of completeness, and also because the author feels that the existence (at least) of the techniques demonstrated is worth knowing.

The main subject-matter of the course continues on p. 6 below.

GRADIENT IN GENERAL COORDINATE SYSTEMS. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a differentiable function (one can think of $n = 2$ or $n = 3$ if one likes). The *gradient* of f is defined to be the vector ∇f in $\mathbf{R}^n$ such that, for any unit vector $\mathbf{n}$, the rate of change of f in the direction $\mathbf{n}$ is equal to $\mathbf{n} \cdot \nabla f$; in other words, such that

$$\lim_{h\to 0} \frac{f(\mathbf{x} + h\mathbf{n}) - f(\mathbf{x})}{h} = \mathbf{n} \cdot \nabla f(\mathbf{x}). \tag{1}$$

In rectangular coordinates in $\mathbf{R}^3$, the gradient has the well-known expression

$$\nabla f(\mathbf{x}) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Now fix some point $\mathbf{x} \in \mathbf{R}^n$ and suppose that $\boldsymbol{\gamma} : (-\epsilon, \epsilon) \to \mathbf{R}^n$ (for some $\epsilon > 0$) is such that $\boldsymbol{\gamma}(0) = \mathbf{x}$, $\boldsymbol{\gamma}'(0) = \mathbf{n}$ (where $\boldsymbol{\gamma}'$ denotes the derivative of $\boldsymbol{\gamma}$ with respect to its parameter). Then by the chain rule we see that

$$\frac{d}{dt} f(\boldsymbol{\gamma}(t))\Big|_{t=0} = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\Big|_{\mathbf{x}} \frac{d\gamma^i}{dt}\Big|_{t=0} = \boldsymbol{\gamma}'(0) \cdot \nabla f(\mathbf{x}) = \mathbf{n} \cdot \nabla f(\mathbf{x});$$

in other words, to determine $\mathbf{n} \cdot \nabla f(\mathbf{x})$, we do not need to use the straight-line path in the definition in (1) above; differentiating along any other curve which passes through the point in the correct direction with unit speed (i.e., satisfying $\boldsymbol{\gamma}'(0) = \mathbf{n}$; unit speed means that $|\boldsymbol{\gamma}'(0)| = |\mathbf{n}| = 1$) will also do.

In particular, let us consider how to express the gradient in curvilinear coordinates. Suppose that $y^1, \ldots, y^n$ is a set of coordinates on some (open) subset of $\mathbf{R}^n$ – this means that we have two sets of functions (letting $x^1, \ldots, x^n$ denote the standard coordinates on $\mathbf{R}^n$)

$$\begin{aligned} y^1 &= y^1(x^1, \ldots, x^n), & x^1 &= x^1(y^1, \ldots, y^n), \\ y^2 &= y^2(x^1, \ldots, x^n), & x^2 &= x^2(y^1, \ldots, y^n), \\ &\vdots & &\vdots \\ y^n &= y^n(x^1, \ldots, x^n), & x^n &= x^n(y^1, \ldots, y^n); \end{aligned}$$

if we think of spherical coordinates on $\mathbf{R}^3$, for example (and readers who feel uncomfortable with the level of generality are highly advised to think only of spherical or cylindrical coordinates in the following), we have

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2}, & & \\ \theta &= \arctan \frac{\sqrt{x^2 + y^2}}{z}, & x &= r \sin\theta \cos\phi, \\ & & y &= r \sin\theta \sin\phi, \\ \phi &= \pm \arctan \frac{y}{x}, & z &= r\cos\theta, \end{aligned}$$

where the $\pm$ in the equation for ϕ is the normal ambiguity in determining ϕ from the ratio $\frac{y}{x}$.

Let us now fix some point $\mathbf{x}_0 \in \mathbf{R}^n$ which has coordinates $(y_0^1, y_0^2, \ldots, y_0^n)$. Now for each $j = 1, 2, \ldots, n$ we have the curve

$$\boldsymbol{\gamma}_j(t) = \left(x^1(y_0^1, \ldots, y_0^j + t, \ldots, y_0^n), x^2(y_0^1, \ldots, y_0^j + t, \ldots, y_0^n), \ldots, x^n(y_0^1, \ldots, y_0^j + t, \ldots, y_0^n)\right),$$

which is just the curve obtained by holding all but the jth coordinate constant and letting the jth coordinate change at unit speed. The unit tangent vector to this curve at $t = 0$, $\frac{\boldsymbol{\gamma}_j'(0)}{|\boldsymbol{\gamma}_j'(0)|}$, is called the *unit coordinate vector in the jth direction at the point* $\mathbf{x}$; we denote it by $\mathbf{y}_j$. It is not hard to see that the vector $\mathbf{y}_j$ is the unit normal to the surface $y^j = y_0^j$ passing through the point $\mathbf{x}$. *Calculating the gradient* in the y coordinate system means representing ∇f in the basis $\{\mathbf{y}_j\}$ at each point. For simplicity in these calculations, we shall when convenient reparametrise the above curves by arclength and let $\boldsymbol{\gamma}_j(s)$ denote the j curve parametrised by arclength $s(t) = \int_0^t |\boldsymbol{\gamma}_j'(t')| dt'$; then we have simply $\mathbf{y}_j = \frac{d\boldsymbol{\gamma}_j}{ds}$.

For example, in spherical coordinates we have the three curves and unit vectors

$$\begin{aligned}
\boldsymbol{\gamma}_1(t) &= ((r_0 + t)\sin\theta_0\cos\phi_0, (r_0 + t)\sin\theta_0\sin\phi_0, (r_0 + t)\cos\theta_0) & \mathbf{r} &= \sin\theta_0\cos\phi_0\mathbf{i} + \sin\theta_0\sin\phi_0\mathbf{j} + \cos\theta_0\mathbf{k}\\
\boldsymbol{\gamma}_2(t) &= (r_0\sin(\theta_0 + t)\cos\phi_0, r_0\sin(\theta_0 + t)\sin\phi_0, r_0\cos(\theta_0 + t)) & \boldsymbol{\theta} &= \cos\theta_0\cos\phi_0\mathbf{i} + \cos\theta_0\sin\phi_0\mathbf{j} - \sin\theta_0\mathbf{k}\\
\boldsymbol{\gamma}_3(t) &= (r_0\sin\theta_0\cos(\phi_0 + t), r_0\sin\theta_0\sin(\phi_0 + t), r_0\cos\theta_0) & \boldsymbol{\phi} &= -\sin\phi_0\mathbf{i} + \cos\phi_0\mathbf{j}
\end{aligned}$$

and the reparametrisation by arclength can be obtained by noting that $\gamma_1(t) = r_0\mathbf{r} + t\mathbf{r}$, and hence is already parametrised by arclength; that $\gamma_2(t)$ represents a circle of radius r_0, so an arclength parameter is $s = r_0 t$; and that $\gamma_3(t)$ represents a circle of radius $r_0\sin\theta_0$, so that an arclength parameter is $s = r_0\sin\theta_0 t$, so that finally we have the parametrisations by arclength –

$$\begin{aligned}
\boldsymbol{\gamma}_1(s) &= ((r_0 + s)\sin\theta_0\cos\phi_0, (r_0 + s)\sin\theta_0\sin\phi_0, (r_0 + s)\cos\theta_0)\\
\boldsymbol{\gamma}_2(s) &= \left(r_0\sin\left(\theta_0 + \frac{s}{r_0}\right)\cos\phi_0, r_0\sin\left(\theta_0 + \frac{s}{r_0}\right)\sin\phi_0, r_0\cos\left(\theta_0 + \frac{s}{r_0}\right)\right)\\
\boldsymbol{\gamma}_3(s) &= \left(r_0\sin\theta_0\cos\left(\phi_0 + \frac{s}{r_0\sin\theta_0}\right), r_0\sin\theta_0\sin\left(\phi_0 + \frac{s}{r_0\sin\theta_0}\right), r_0\cos\theta_0\right).
\end{aligned}$$

The vectors $\{\mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\phi}\}$ are seen to give an orthonormal basis for $\mathbf{R}^3$ for any values of θ_0 and ϕ_0.

Returning to our general picture, let us now assume that (as for the case of spherical and – it can be shewn – cylindrical coordinates) the vectors $\mathbf{y}_j$ are all mutually orthogonal (and hence orthonormal since they have unit length by construction). Then we have simply

$$\nabla f(\mathbf{x}_0) = (\mathbf{y}_1 \cdot \nabla f(\mathbf{x}_0))\,\mathbf{y}_1 + \cdots + (\mathbf{y}_n \cdot \nabla f(\mathbf{x}_0))\,\mathbf{y}_n.$$

Now by our work above, we have (since by the definition of arclength, we have $\frac{ds}{dt} = |\boldsymbol{\gamma}_j'|$, so $\frac{dt}{ds} = \frac{1}{|\boldsymbol{\gamma}_j'|}$)

$$\begin{aligned}
\mathbf{y}_j \cdot \nabla f(\mathbf{x}_0) &= \frac{d}{ds}\left(f(\boldsymbol{\gamma}_j(s))\right)\Big|_{s=0}\\
&= \frac{d}{dt}\left(f(\boldsymbol{\gamma}_j(t))\right)\Big|_{t=0}\frac{dt}{ds}\Big|_{s=0} = \frac{1}{|\boldsymbol{\gamma}_j'(0)|}\frac{d}{dt}\left(f(\boldsymbol{\gamma}_j(t))\right)\Big|_{t=0}\\
&= \frac{1}{|\boldsymbol{\gamma}_j'(0)|}\sum_{i=1}^n \frac{\partial f}{\partial x^i}\frac{d\boldsymbol{\gamma}_j^i}{dt}\Big|_{t=0}\\
&= \frac{1}{|\boldsymbol{\gamma}_j'(0)|}\sum_{i=1}^n \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial y^j}\Big|_{\mathbf{x}=\mathbf{x}_0}\\
&= \frac{1}{|\boldsymbol{\gamma}_j'(0)|}\frac{\partial f}{\partial y^j}\Big|_{(y_0^i)}.
\end{aligned}$$

Applying this formula to the special case of spherical coordinates, we see first of all that (the derivatives are with respect to t, not s)

$$|\gamma_1'(0)| = 1, \qquad |\gamma_2'(0)| = r_0, \qquad |\gamma_3'(0)| = r_0 \sin\theta_0.$$

Thus we obtain

$$\begin{aligned}
\mathbf{y}_1 \cdot \nabla f(\mathbf{x}_0) &= \mathbf{r} \cdot \nabla f(\mathbf{x}_0) = \frac{\partial f}{\partial r}, \\
\mathbf{y}_2 \cdot \nabla f(\mathbf{x}_0) &= \boldsymbol{\theta} \cdot \nabla f(\mathbf{x}_0) = \frac{1}{r}\frac{\partial f}{\partial \theta}, \\
\mathbf{y}_3 \cdot \nabla f(\mathbf{x}_0) &= \boldsymbol{\phi} \cdot \nabla f(\mathbf{x}_0) = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi},
\end{aligned}$$

where all quantities are to be evaluated at the point (r_0, θ_0, ϕ_0). Thus we have finally

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{\phi}.$$

Similarly, in cylindrical coordinates we have the three curves and unit vectors

$$\begin{aligned}
\gamma_1(t) &= ((\rho_0 + t)\cos\phi_0, (\rho_0 + t)\sin\phi_0, z) & \boldsymbol{\rho} &= \cos\phi_0 \mathbf{i} + \sin\phi_0 \mathbf{j} \\
\gamma_2(t) &= (\rho_0 \cos(\phi_0 + t), \rho_0 \sin(\phi_0 + t), z) & \boldsymbol{\phi} &= -\sin\phi_0 \mathbf{i} + \cos\phi_0 \mathbf{j} \\
\gamma_3(t) &= (\rho_0 \cos\phi_0, \rho_0 \sin\phi_0, z + t) & \mathbf{z} &= \mathbf{k}
\end{aligned}$$

and

$$|\gamma_1'(0)| = 1, \qquad |\gamma_2'(0)| = \rho_0, \qquad |\gamma_3'(0)| = 1,$$

so that

$$\nabla f = \frac{\partial f}{\partial \rho}\boldsymbol{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\boldsymbol{\phi} + \frac{\partial f}{\partial z}\mathbf{k}.$$

DIVERGENCE. For this section we shall work exclusively in $\mathbf{R}^3$. Recall that the *divergence* of a vector field $\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}$ in $\mathbf{R}^3$ is defined by

$$\operatorname{div} \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z},$$

and that we have the *divergence theorem*

$$\iiint_V \operatorname{div} \mathbf{F} dV = \iint_{\partial V} \mathbf{F} \cdot \mathbf{n} dS,$$

where $\mathbf{n}$ represents the outwards unit normal to the boundary ∂V of V.

We also note for future reference that, given a general coordinate system $\{y^j\}$ as above, the *area element* in a surface of constant coordinate y^j is given by

$$A_j := |\gamma_i' \times \gamma_k'|,$$

where i and k are the two elements of $\{1, 2, 3\}$ not equal to j. Thus, for example, in the case of spherical coordinates (recalling the formula $|\mathbf{A} \times \mathbf{B}| = |A||B|\sin\theta_{\mathbf{AB}}$, where $\theta_{\mathbf{AB}}$ is the angle between $\mathbf{A}$ and $\mathbf{B}$, and that the vectors γ_j' are all mutually orthogonal so $\sin\theta_{\gamma_i'\gamma_k'} = 1$ for all i and k, so that $|\gamma_i' \times \gamma_j'| = |\gamma_i'||\gamma_j'|$; this formula makes sense when we consider that we are taking the area of a small rectangle whose sides have length $|\gamma_i'|$ and $|\gamma_j'|$), the area elements in surfaces of constant r, θ, and ϕ are given respectively by

$$\begin{aligned}
|\gamma_2' \times \gamma_3'| &= r^2 |\cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k}|| - \sin\theta\sin\phi\mathbf{i} + \sin\theta\cos\phi\mathbf{j}| \\
&= r^2 \sin\theta, \\
|\gamma_1' \times \gamma_3'| &= |\sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}|| - r\sin\theta\sin\phi\mathbf{i} + r\sin\theta\cos\phi\mathbf{j}| \\
&= r\sin\theta, \\
|\gamma_1' \times \gamma_2'| &= |\sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}||r\cos\theta\cos\phi\mathbf{i} + r\cos\theta\sin\phi\mathbf{j} - r\sin\theta\mathbf{k}| \\
&= r.
\end{aligned}$$

Let us now return to the case of a general coordinate system, but still assume it to be orthogonal (meaning that the vectors $\mathbf{y}_j$ are mutually orthogonal at all points of $\mathbf{R}^3$), pick some point $\mathbf{x}_0 \in \mathbf{R}^3$ with coordinates (y_0^i), and apply the divergence theorem to the small curvilinear cube given by

$$V = [y_0^1, y_0^1 + \Delta y^1] \times [y_0^2, y_0^2 + \Delta y^2] \times [y_0^3, y_0^3 + \Delta y^3].$$

Then, by the change-of-variables formula and the mean value theorem for integrals, there will be some point (y_*^i) in this cube such that

$$\iiint_V \operatorname{div} \mathbf{F} dV = \operatorname{div} \mathbf{F}(y_*^i) J \Delta y^1 \Delta y^2 \Delta y^3,$$

where J is the Jacobian of the coordinate transformation $\mathbf{x} \mapsto \mathbf{y}$; we note that $J = |\boldsymbol{\gamma}_1 \cdot (\boldsymbol{\gamma}_2 \times \boldsymbol{\gamma}_3)| = |\boldsymbol{\gamma}_1||\boldsymbol{\gamma}_2||\boldsymbol{\gamma}_3|$, since the vectors are all orthogonal.

Let us now consider the right-hand side of the divergence theorem. The cube given above has evidently six faces; these can be grouped into three pairs, the treatment of each of which is analogous. Let us work with the pair

$$\{y_0^1\} \times [y_0^2, y_0^2 + \Delta y^2] \times [y_0^3, y_0^3 + \Delta y^3] \cup \{y_0^1 + \Delta y^1\} \times [y_0^2, y_0^2 + \Delta y^2] \times [y_0^3, y_0^3 + \Delta y^3].$$

The unit normal vector on the second part of this pair will simply be the vector $\mathbf{y}_1$, while that on the first will be (since we need the *outer* normal in the divergence theorem) $-\mathbf{y}_1$; thus the integral on the right-hand side of the divergence theorem corresponding to these two surfaces is equal to (we let $F^j = \mathbf{y}^j \cdot \mathbf{F}$)

$$\begin{aligned}
&\int_{y_0^2}^{y_0^2+\Delta y^2} \int_{y_0^3}^{y_0^3+\Delta y^3} F^1(y_0^1 + \Delta y^1, y^2, y^3) A_1(y_0^1 + \Delta y^1, y^2, y^3) dy^3 dy^2 \\
&\qquad\qquad - \int_{y_0^2}^{y_0^2+\Delta y^2} \int_{y_0^3}^{y_0^3+\Delta y^3} F^1(y_0^1, y^2, y^3) A_1(y_0^1, y^2, y^3) dy^3 dy^2 \\
&= \int_{y_0^2}^{y_0^2+\Delta y^2} \int_{y_0^3}^{y_0^3+\Delta y^3} \left(F^1 A_1\right)(y_0^1 + \Delta y^1, y^2, y^3) - \left(F^1 A_1\right)(y_0^1, y^2, y^3) dy^2 dy^3 \\
&= \int_{y_0^2}^{y_0^2+\Delta y^2} \int_{y_0^3}^{y_0^3+\Delta y^3} \left.\frac{\partial\left(F^1 A_1\right)}{\partial y^1}\right|_{(y_0^1, y^2, y^3)} \Delta y^1 + o(\Delta y^1) dy^2 dy^3 \\
&= \left(\left.\frac{\partial\left(F^1 A_1\right)}{\partial y^1}\right|_{(y_0^1, y_*^2, y_*^3)} \Delta y^1 + o(\Delta y^1) \right) \Delta y^2 \Delta y^3,
\end{aligned}$$

where $o(h)$ represents a quantity which satisfies

$$\lim_{h \to 0} \frac{o(h)}{h} = 0,$$

and we have again used the mean value theorem for integrals. (Here, and below, in order to keep the notation from becoming too cumbersome we shall use (y_*^i) to denote any point that lies in the above cube; it may represent multiple different points on the same line. This will not ultimately cause any troubles since we will take a limit which forces $(y_*^i) \to (y_0^i)$ at the end.) The other two pairs are treated similarly, giving rise finally to the equation

$$\begin{aligned}
\operatorname{div} \mathbf{F}(y_*^i) J \Delta y^1 \Delta y^2 \Delta y^3 = &\left(\left.\frac{\partial\left(F^1 A_1\right)}{\partial y^1}\right|_{(y_0^1, y_*^2, y_*^3)} \Delta y^1 + o(\Delta y^1) \right) \Delta y^2 \Delta y^3 \\
&+ \left(\left.\frac{\partial\left(F^2 A_2\right)}{\partial y^2}\right|_{(y_*^1, y_0^2, y_*^3)} \Delta y^2 + o(\Delta y^2) \right) \Delta y^1 \Delta y^3 \\
&+ \left(\left.\frac{\partial\left(F^3 A_3\right)}{\partial y^3}\right|_{(y_*^1, y_*^2, y_0^3)} \Delta y^3 + o(\Delta y^3) \right) \Delta y^1 \Delta y^2.
\end{aligned}$$

If we now divide through by $J\Delta y^1 \Delta y^2 \Delta y^3$ and take the limit as $\Delta y^1, \Delta y^2, \Delta y^3 \to 0$,[1] we obtain finally the expression (since all points (y_*^i) must go to (y_0^i) in this limit)

$$\operatorname{div} \mathbf{F} = \frac{1}{J}\left(\frac{\partial\left(F^1 A_1\right)}{\partial y^1} + \frac{\partial\left(F^2 A_2\right)}{\partial y^2} + \frac{\partial\left(F^3 A_3\right)}{\partial y^3}\right).$$

In particular, in spherical coordinates we have

$$J = r^2 \sin\theta, \qquad A_1 = r^2 \sin\theta, \qquad A_2 = r \sin\theta, \qquad A_3 = r,$$

whence we obtain (writing $\mathbf{F} = F_r \mathbf{r} + F_\theta \boldsymbol{\theta} + F_\phi \boldsymbol{\phi}$)

$$\begin{aligned}
\operatorname{div} \mathbf{F} &= \frac{1}{r^2 \sin\theta}\left(\frac{\partial\left(r^2 \sin\theta F_r\right)}{\partial r} + \frac{\partial\left(r \sin\theta F_\theta\right)}{\partial\theta} + \frac{\partial\left(r F_\phi\right)}{\partial\phi}\right) \\
&= \frac{\partial F_r}{\partial r} + \frac{2}{r} F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{1}{r}\cot\theta F_\theta + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}.
\end{aligned}$$

Similarly, for cylindrical coordinates we have the area elements

$$\begin{aligned}
A_1 &= |-\rho \sin\phi \mathbf{i} + \rho\cos\phi \mathbf{j}||\mathbf{k}| = \rho, \\
A_2 &= |\cos\phi \mathbf{i} + \sin\phi \mathbf{j}||\mathbf{k}| = 1, \\
A_3 &= |\cos\phi \mathbf{i} + \sin\phi \mathbf{j}||-\rho\sin\phi \mathbf{i} + \rho\cos\phi \mathbf{j}| = \rho,
\end{aligned}$$

while $J = r$; thus we have the formula (writing $\mathbf{F} = F_\rho \boldsymbol{\rho} + F_\phi \boldsymbol{\phi} + F_z \mathbf{k}$)

$$\begin{aligned}
\operatorname{div} \mathbf{F} &= \frac{1}{\rho}\left(\frac{\partial\left(\rho F_\rho\right)}{\partial\rho} + \frac{\partial F_\phi}{\partial\phi} + \frac{\partial\left(\rho F_z\right)}{\partial z}\right) \\
&= \frac{\partial F_\rho}{\partial\rho} + \frac{1}{\rho} F_\rho + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}.
\end{aligned}$$

Finally, putting all of this together with the expressions for the gradients derived above gives the following expressions for the Laplacian in spherical and cylindrical coordinates:

$$\begin{aligned}
\nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}, \\
\nabla^2 u &= \frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\partial^2 u}{\partial z^2}.
\end{aligned}$$

SEPARATION OF VARIABLES IN SPHERICAL COORDINATES. Consider now Laplace's equation in spherical coordinates,

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0. \tag{2}$$

As we did when treating Laplace's equation in rectangular coordinates, we begin by seeking simple solutions of the form

$$u = R(r)\Theta(\theta)\Phi(\phi),$$

[1] Note that there is one other subtle point which must be dealt with here, namely whether the quantities $\frac{o(\Delta y^1)}{\Delta y^1}$ etc. go to zero uniformly in the other Δy^i. They will if we assume that the vector field $\mathbf{F}$ possesses continuous second-order derivatives.

in the hopes that the general solution can be expressed in a series of such solutions. Substituting this into equation (2) and dividing by u, we obtain (here prime denotes differentiation with respect to the whatever single variable the function depends on; e.g., $R' = \frac{dR}{dr}$)

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\left(\frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} + \frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi}\right) = 0.$$

Now we see that of all the terms on the left-hand side, only $\frac{\Phi''}{\Phi}$ depends on ϕ; hence it must be constant. (Somewhat more explicitly, note that we may solve the above equation for $\frac{\Phi''}{\Phi}$, obtaining

$$\frac{\Phi''}{\Phi} = -\sin^2\theta\left(r^2\frac{R''}{R} + 2r\frac{R'}{R} + \frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta}\right);$$

now the right-hand side of the above expression does not depend on ϕ, and hence neither can the left-hand side, i.e., $\frac{\Phi''}{\Phi}$ is constant, as claimed.) We would like to know something about this constant before proceeding further. Suppose that we are interested in solving Laplace's equation on a ball (the interior of a sphere): then the solution must be valid, continuous, and single-valued for all values of the angle ϕ. Since increasing ϕ by 2π leaves us at the same point, out solution must be periodic in ϕ with angle 2π. Since Φ is the only part of the solution depending on ϕ, this means that Φ must itself be periodic with period 2π. Now we know that if $\frac{\Phi''}{\Phi}$ is positive, then Φ will be a linear combination of sinh and cosh, and hence will *not* be periodic; thus $\frac{\Phi''}{\Phi}$ must be zero or negative. If it is zero, then it must be of the form $a + b\phi$; again, ϕ is not periodic, and hence we must have $b = 0$, i.e., in this case Φ must be a constant. (This corresponds to what is called an *azimuthally symmetric* solution; we shall have more to say about this when we discuss Legendre's equation and Legendre polynomials shortly.) Otherwise, $\frac{\Phi''}{\Phi}$ must be negative, and we may write it as $-m^2$ for some positive real number m. (Choosing $m > 0$ is simply a convention; we could as well have chosen $m < 0$; but we cannot have both. Here we choose $m > 0$.) Thus $\Phi'' = -m^2\Phi$, which has as a general solution $\Phi_m = a_m\cos m\phi + b_m\sin m\phi$. Since Φ_m must have period 2π (general periodicity is not enough), we must actually have $m \in Z$. Thus the ϕ dependence of our solution will be of the form $a_m\cos m\phi + b_m\sin m\phi$ (note that we could also have used the complex basis $e^{im\phi}$).[2]

Substituting $\frac{\Phi''}{\Phi} = -m^2$ back into Laplace's equation, and multiplying by r^2, we obtain

$$r^2\frac{R''}{R} + 2r\frac{R'}{R} + \left(\frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} - \frac{m^2}{\sin^2\theta}\right) = 0.$$

Again, the first of these two terms depends only on R, and the second depends only on Θ, which means (as with Φ) that each of them must be constant. Let us let α[3] denote the term in parentheses, so that we obtain for R the equation

$$r^2\frac{R''}{R} + 2r\frac{R'}{R} = -\alpha,$$

or

$$r^2R'' + 2rR' + \alpha R = 0.$$

[2]As hinted above, and mentioned in somewhat greater detail in class, this form for Φ is contingent on the region over which we are solving containing a full range of angles ϕ. Should we be solving only on a wedge, for example, then not only would we no longer necessarily have $m \in Z$, we might actually need to consider also the exponential solutions for Φ – at least in principle. In this case, we would need boundary conditions on the constant-ϕ boundaries, much as we have boundary conditions on the constant-y and constant-x boundaries in the problems we have done in rectangular coordinates. For the moment, though, to keep the discussion simple, we shall stick with this form for Φ.

[3]It would be more natural to denote this constant by $-\alpha$, but since the author was careless and denoted it by α in the lecture, it seems prudent to keep that convention here. At any rate, as noted in lecture and as will be pointed out shortly, α itself is not really the fundamental quantity; ℓ is much more fundamental.

The form of this equation suggests that it should possess power-law solutions; thus let us try to find solutions of the form $R = r^\ell$, for some ℓ (which at present we know nothing about). Substituting this expression in for R, we obtain

$$\begin{aligned} r^2\left(\ell(\ell-1)r^{\ell-2}\right) + 2r\left(\ell r^{\ell-1}\right) + \alpha r^\ell &= \ell(\ell-1)r^\ell + 2\ell r^\ell + \alpha r^\ell \\ &= \left[\ell(\ell+1) + \alpha\right] r^\ell = 0, \end{aligned}$$

from which we see that ℓ must satisfy the equation $\ell(\ell+1) = -\alpha$. This is a quadratic equation with solutions

$$\ell = -\frac{1}{2} \pm \frac{1}{2}(1-4\alpha)^{\frac{1}{2}}.$$

(Note that these may be complex.) From this we obtain also the result that if ℓ is one solution to $\ell(\ell+1) = -\alpha$, then $-(\ell+1)$ is the other solution. Thus in general we have the solution

$$R_\ell = a_\ell r^\ell + b_\ell r^{-(\ell+1)}.$$

Repeated roots occur when $\alpha = \frac{1}{4}$; and if $\alpha > \frac{1}{4}$ the roots will be complex: while the expressions r^ℓ and $r^{-(\ell+1)}$ can still be defined in this case, they are not as simple. For reasons which shall become apparent when we study Legendre's equation in a moment, we are interested mostly in cases in which ℓ is a nonnegative integer. Thus (as with our choice for m above) we shall for the moment restrict to this case. Thus we consider only α which are of the form $-\ell(\ell+1)$ for some $\ell \in \mathbf{Z}$, $\ell \geq 0$. (It is because of this that we said above that ℓ is more fundamental than α, so that our use of α instead of $-\alpha$ was not that important.)

Having solved the equations for Φ and R, let us now treat the equation for Θ. This is the most interesting of them all and will introduce us to the field of *orthogonal polynomials* through the so-called Legendre polynomials.

Setting $\alpha = -\ell(\ell+1)$, we see that we obtain for Θ the equation

$$\Theta'' + \cot\theta\,\Theta' + \left(\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right)\Theta = 0. \tag{3}$$

Unfortunately, as it stands there is no clear way to approach this equation, since while it is a second-order linear ordinary differential equation it has variable coefficients. It turns out to be useful to make the change of variables $x = \cos\theta$ (here x does *not* refer to the Cartesian coordinate corresponding to the spherical coordinate system we are using – that would be $r\sin\theta\cos\phi$); note that this implies that $x \in [-1, 1]$. For this change of variables, the chain rule gives (for some function f)

$$\begin{aligned} \frac{df}{d\theta} &= \frac{df}{dx}\frac{dx}{d\theta} = -\sin\theta\frac{df}{dx}, \\ \frac{d^2f}{d\theta^2} &= \frac{d}{d\theta}\left(-\sin\theta\frac{df}{dx}\right) \\ &= -\cos\theta\frac{df}{dx} - \sin\theta\left(-\sin\theta\frac{d^2f}{dx^2}\right) = -x\frac{df}{dx} + (1-x^2)\frac{d^2f}{dx^2}, \end{aligned}$$

whence we see that equation (3) becomes, letting $P(x)$ be the function of x corresponding to $\Theta(\theta)$ (and since $\cot\theta\sin\theta = \cos\theta = x$ in the second term in that equation)

$$(1-x^2)\frac{d^2P}{dx^2} - x\frac{dP}{dx} - x\frac{dP}{dx} + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P = (1-x^2)P'' - 2xP' + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P = 0.$$

This equation is called *Legendre's equation*, and the solutions for nonnegative integers ℓ and m are called the *associated Legendre functions*. Since $x = \cos\theta \in [-1, 1]$, it is an equation on $[-1, 1]$.

Let us consider the special case $m = 0$; in this case there is no ϕ dependence and our solution is azimuthally symmetric. The equation in this case is simply

$$(1-x^2)P'' - 2xP' + \ell(\ell+1)P = 0. \tag{4}$$

We shall look for a solution P which has a power series expansion around $x = 0$; in other words, we look for a solution

$$P = \sum_{n=0}^{\infty} a_n x^n.$$

Substituting this expression in to the above equation, we obtain

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} (1-x^2) a_n n(n-1) x^{n-2} - 2x n a_n x^{n-1} + \ell(\ell+1) a_n x^n \\
&= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - a_n n(n-1)x^n - 2n a_n x^n + \ell(\ell+1) a_n x^n \\
&= \sum_{n=0}^{\infty} \left(a_{n+2}(n+2)(n+1) + a_n \left(\ell(\ell+1) - n(n+1)\right)\right) x^n,
\end{aligned}$$

from which we obtain the recurrence relation

$$a_{n+2} = a_n \frac{n(n+1) - \ell(\ell+1)}{(n+2)(n+1)}.$$

We see that this will determine all even coefficients a_{2k} given a_0, and all odd coefficients a_{2k+1} given a_1; since we started with a second-order differential equation, it is natural that we have two undetermined coefficients. (Another way of looking at it is to think of a_0 and a_1 as being the coefficients in the linear combination giving the general solution to the equation.) Moreover, if $a_0 = 0$, then all even coefficients will vanish, and if $a_1 = 0$ then all odd coefficients will vanish.

We note something else about this recurrence relation: If $n(n+1) = \ell(\ell+1)$ for some n, then a_{n+2} and hence a_{n+2k} for all $k > 0$ will vanish. This means that if $n(n+1) = \ell(\ell+1)$ for some odd integer n, then there will be only finitely many odd-power terms in the power series, while if $n(n+1) = \ell(\ell+1)$ for some even integer n there will be only finitely many even-power terms in the power series. In either case, by requiring the terms of opposite valence to vanish (i.e., setting $a_0 = 0$ in the first case and $a_1 = 0$ in the second case), we obtain power series solutions which are *finite* – which is to say, polynomial solutions. These are called the *Legendre polynomials*.

Let us be more specific. Suppose that $\ell = 2k$ for some $k \in \mathbf{Z}$, $k \geq 0$, and let $a_1 = 0$. Then, as noted above, all odd coefficients in the series will vanish. Moreover,

$$a_{2k+2} = a_{2k} \frac{2k(2k+1) - \ell(\ell+1)}{(2k+2)(2k+1)} = a_{2k} \frac{2k(2k+1) - 2k(2k+1)}{(2k+2)(2k+1)} = 0,$$

and thus $a_{2k+2j} = 0$ for all $j \in \mathbf{Z}$, $j > 0$. Since all odd-order coefficients vanish, the power series will truncate and we will be left with a polynomial of degree $2k = \ell$.

Similarly, suppose that $\ell = 2k+1$ for some $k \in \mathbf{Z}$, $k \geq 0$, and let now $a_0 = 0$. Then all even terms vanish; moreover, as before,

$$a_{2k+3} = a_{2k+1} \frac{(2k+1)(2k+2) - \ell(\ell+1)}{(2k+3)(2k+2)} = a_{2k+1} \frac{(2k+1)(2k+2) - (2k+1)(2k+2)}{(2k+3)(2k+2)} = 0,$$

so $a_{2k+1+2j} = 0$ for all $j \in \mathbf{Z}$, $j > 0$, and our power series truncates to give a polynomial of order $2k+1 = \ell$.

Thus we see that whenever ℓ is a nonnegative integer, equation (4) will have a solution which is a polynomial of degree ℓ. It is determined up to an overall multiplicative factor. We denote by $P_\ell(x)$ the polynomial satisfying (4) and satisfying also $P_\ell(1) = 1$; this will fix the value of a_0 (ℓ even) or a_1 (ℓ odd), which we left open above. $P_\ell(x)$ is called the *ℓth Legendre polynomial*, or the *Legendre polynomial of degree ℓ*.[4]

[4]Since our original equation was second-order, even in the case where ℓ is a nonnegative integer it will possess another solution linearly independent of $P_\ell(x)$; this would correspond to letting the other one of a_0 or a_1 equal something nonzero. Since it turns out that the set of Legendre polynomials is complete on the interval $[-1, 1]$, they are sufficient for our purposes at the moment.

EXAMPLES. Let us compute the first few Legendre polynomials. If $\ell = 0$, we seek a polynomial of degree 0, i.e., a constant polynomial, with $P_0(1) = 1$; thus $P_0(x) = 1$ for all x. If $\ell = 1$, then we set $a_0 = 0$ and leave a_1 undetermined for the moment; but then $a_3 = 0$, so $P_1(x) = a_1 x$ and the normalisation condition $P_1(1) = 1$ implies that $a_1 = 1$.

The case $\ell = 2$ is a bit more interesting. In this case we set $a_1 = 1$ and leave a_0 undetermined; then we have

$$a_2 = a_0 \frac{0(0+1) - 2(2+1)}{(0+2)(0+1)} = -3a_0,$$

while a_4 and all higher-order coefficients vanish. Thus $P_2(x) = -3a_0 x^2 + a_0$, so $P_2(1) = -2a_0 = 1$ forces $a_0 = -\frac{1}{2}$ and $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$.

EXAMPLES OF SOLUTIONS TO LAPLACE'S EQUATION. Let us see how all of these results may be pulled together to give some simple solutions to Laplace's equation on the unit sphere.

(a) Solve the boundary-value problem on the unit boll $\{(r, \theta, \phi) | r < 1\}$:

$$\nabla^2 u = 0, \qquad u|_{r=1} = 1.$$

Since the boundary data and the region are both spherically symmetric, we anticipate that the solution will be as well, meaning that we expect a solution depending only on r; this is equivalent to looking for a separated solution with Θ and Φ both constant, which means (in the context of what we have just done) that $m = \ell = 0$. In this case, the equation for R becomes simply

$$r^2 R'' + 2rR' = 0,$$

and by our previous work this has general solution $R = a + \frac{b}{r}$, and this will also be the form of our solution u. Since we wish u to satisfy $\nabla^2 u = 0$ everywhere on the interior of the unit sphere, u must in particular be continuous and finite there, and thus we must have $b = 0$, so $u = a$ is just a constant. The boundary condition then gives $a = 1$, so the solution to this boundary-value problem is simply $u = 1$. (We could also have obtained this by inspection.)

(b) Solve the boundary-value problem on the set $\{(r, \theta, \phi) | 1 < r < 2\}$:

$$\nabla^2 u = 0, \qquad u|_{r=1} = 1, \quad u|_{r=2} = 0.$$

In this case we still have a spherically symmetric region and spherically symmetric boundary data, so we expect to obtain a spherically symmetric solution. By our work in part (a), we see immediately that we must have $u = a + \frac{b}{r}$ for some constants a, b. In this case we can no longer immediately set $b = 0$ since the point $r = 0$ (which is where the second term goes to infinity) is not in the region where we require $\nabla^2 u = 0$. This allows us to fit both boundary conditions, as follows. We have

$$\begin{aligned} u|_{r=1} &= a + b = 1 \\ u|_{r+2} &= a + \frac{b}{2} = 0, \end{aligned}$$

from which we see easily that $b = 2$, $a = -1$, so $u = -1 + \frac{2}{r}$ is the solution to the boundary-value problem.

(c) Solve the boundary-value problem on the unit ball:

$$\nabla^2 u = 0, \qquad u|_{r=1} = \cos\theta.$$

In this case we no longer have spherical symmetry, though we do have azimuthal symmetry, meaning that our solution will not depend on ϕ. In general, our approach to solving this type of problem is very similar to our approach for solving boundary-value problems on a square: we suppose that our solution can be written as a series of separated solutions, in this case

$$u(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \left(a_\ell r^\ell + b_\ell r^{-(\ell+1)} \right) P_\ell(\cos\theta);$$

in the present case, as in (a), since we wish u to satisfy Laplace's equation on the unit ball, we must set all of the b_ℓ to zero (this is similar to how we used the boundary conditions to require that the coefficients of all of the cosine terms vanished when we solved Laplace's equation on the unit square, although the reason is different). We then apply the remaining boundary condition:

$$u(1, \theta, \phi) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(\cos\theta) = \cos\theta,$$

and try to determine a_ℓ. We shall see soon that $\{P_\ell(x) | \ell \in \mathbf{Z}, \ell \geq 0\}$ forms an orthogonal set on $[-1, 1]$; it is also complete (though we shall not prove this at present), and thus for any reasonable boundary data $u(1, \theta)$ it will always be possible to find coefficients a_ℓ satisfying the above equation – and moreover these coefficients will be unique. At present it is sufficient to note that $\cos\theta = P_1(\cos\theta)$, so that we may take simply $a_1 = 1$, $a_\ell = 0$, $\ell \neq 1$ (note that this a_1 is completely different from the a_1 we had above when we investigated the power series representation of P_ℓ!). The final solution is then simply $u = rP_1(\cos\theta) = r\cos\theta = z$.

Another way of looking at the above description of our method is as follows. Suppose that we are solving on the unit ball and given boundary data $P_\ell(\cos\theta)$ on the unit sphere. Then we know that the corresponding radial solution is $ar^\ell + br^{-(\ell+1)}$, but we reject the second term (i.e., set $b = 0$) since this term is not continuous on the unit ball; thus our solution must be of the form $ar^\ell P_\ell(\cos\theta)$, and since our boundary data is $P_\ell(\cos\theta)$ on the unit ball, we must have $a = 1$, and our solution is $r^\ell P_\ell(\cos\theta)$. (Were we given the same boundary data, but on the ball $\{(r, \theta, \phi) | r \leq r_0\}$, then we would need $ar_0^\ell = 1$, so we would set $a = r_0^{-\ell}$ and our solution would be $\left(\frac{r}{r_0}\right)^\ell P_\ell(\cos\theta)$.) If our boundary data is a linear combination (or a series) of P_ℓ for different ℓ, then this method may be applied to each term in the linear combination, and then sum the results to get the full solution. In the case where our boundary data is a series in the P_ℓ, we must use methods of orthogonal functions to determine the coefficients, as we did when solving Laplace's equation in rectangular coordinates. We shall discuss this in more detail later.

The moral of the story is: boundary data $P_\ell(\cos\theta)$ gives rise to a solution of the form

$$\left(ar^\ell + br^{-(\ell+1)}\right) P_\ell(\cos\theta),$$

with a and b to be determined from the other requirements in the problem, and general boundary data may be treated by linearity. This is analogous to how the initial data $\sin kx$ leads to a solution $\sin kx e^{-k^2 Dt}$ to the heat equation, as we discussed in the first week of class, or to how boundary data $\sin n\pi x$ leads to a solution $\sin n\pi x\, (a \sinh n\pi y + b \cosh n\pi y)$ to Laplace's equation on the unit square.

For more complicated problems, such as those on Homework 4, variants and combinations of the above methods may be used.