Summary:

- The temperature in a body satisfies the equation $\frac{\partial u}{\partial t} = D\nabla^2 u$ for some constant D, where (in three dimensions) $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.
- In one dimension, this becomes $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$.
- If we require initial data $u(x, 0) = \sin kx$ for some constant k, then a corresponding solution to this equation is $u(x, t) = \sin kx e^{-k^2 Dt}$.
- Thus, if we require initial data $u(x, 0) = \sum \sin k_n x$, where the sum is over a finite collection of k_n, then a corresponding solution is $u(x, t) = \sum \sin k_n x e^{-k_n^2 Dt}$.
- This can be extended to infinite sums (and even integrals), which will allow us to represent (almost) any initial data on a bounded interval.

NOTATION. We use the notations $\frac{\partial u}{\partial x}$, u_x, and $\partial_x u$ to denote the partial derivative of u with respect to x. They are all equivalent.

EXAMPLE from ordinary differential equations. Let $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, and consider the equation (where a dot indicates differentiation with respect to t)

$$\text{(1)} \qquad \dot{\mathbf{x}} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x}, \qquad \begin{aligned} \dot{x}_1 &= 2x_1 + x_2 \\ \dot{x}_2 &= x_1 + 2x_2 \end{aligned}$$

with initial data $\mathbf{x}(0) = \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix}$. As it stands, this is a coupled system, which is difficult to solve directly. It can be decoupled by diagonalising the coefficient matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, as follows. Let us denote this matrix by A. It has characteristic equation

$$\begin{aligned} 0 &= \det\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3, \end{aligned}$$

which has roots $\lambda = 3$, $\lambda = 1$. We see that

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are corresponding eigenvectors. It is useful to normalise these; thus we set

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

These vectors are clearly linearly independent and hence span $\mathbf{R}^2$; thus for each t there must exist numbers $y_1(t)$, $y_2(t)$ such that

$$\mathbf{x}(t) = y_1(t)\mathbf{e}_1 + y_2(t)\mathbf{e}_2.$$

If we substitute this back in to equation (1) above, we obtain

$$\begin{aligned} \dot{y}_1(t)\mathbf{e}_1 + \dot{y}_2(t)\mathbf{e}_2 &= A\left(y_1(t)\mathbf{e}_1 + y_2(t)\mathbf{e}_2\right) \\ &= y_1(t)A\mathbf{e}_1 + y_2(t)A\mathbf{e}_2 = y_1(t)\left(3\mathbf{e}_1\right) + y_2(t)\mathbf{e}_2 = 3y_1(t)\mathbf{e}_1 + y_2(t)\mathbf{e}_2 \end{aligned}$$

since $\mathbf{e}_1$ and $\mathbf{e}_2$ are eigenvectors of A with eigenvalues 3 and 1, respectively. Thus we obtain the two equations

$$\begin{aligned} \dot{y}_1(t) &= 3y_1(t) \\ \dot{y}_2(t) &= y_2(t) \end{aligned}$$

which are easily solved to give $y_1(t) = y_1(0)e^{3t}$, $y_2(t) = y_2(0)e^t$.What this means is that if our initial data is equal to $\mathbf{e}_1$, so that $y_1(0) = 1$, $y_2(0) = 0$, then our solution will be

$$y_1(t)\mathbf{e}_1 = e^{3t}\mathbf{e}_1,$$

while if our initial data is instead equal to $\mathbf{e}_2$, so that $y_1(0) = 0$, $y_2(0) = 1$, then our solution will be

$$y_2(t)\mathbf{e}_1 = e^t\mathbf{e}_2.$$

In general, our solution will be a linear combination of these, depending on $y_1(0)$ and $y_2(0)$. To find $y_1(0)$ and $y_2(0)$ in terms of $\mathbf{x}(0)$, we may proceed as follows. We have

$$\begin{aligned}\mathbf{x}(0)\cdot\mathbf{e}_1 &= (y_1(0)\mathbf{e}_1 + y_2(0)\mathbf{e}_2)\cdot\mathbf{e}_1\\ &= y_1(0)\mathbf{e}_1\cdot\mathbf{e}_1 + y_2(0)\mathbf{e}_2\cdot\mathbf{e}_1 = y_1(0),\end{aligned}$$

since $\mathbf{e}_1\cdot\mathbf{e}_1 = 1$ and (crucially) $\mathbf{e}_2\cdot\mathbf{e}_1 = 0$. (We see that had we not normalised, this procedure would still work; we would just have to divide $\mathbf{x}(0)\cdot\mathbf{e}_1$ by $\mathbf{e}_1\cdot\mathbf{e}_1$ to find $y_1(0)$.) In exactly the same way, we see that

$$\begin{aligned}\mathbf{x}(0)\cdot\mathbf{e}_2 &= (y_1(0)\mathbf{e}_1 + y_2(0)\mathbf{e}_2)\cdot\mathbf{e}_2\\ &= y_1(0)\mathbf{e}_1\cdot\mathbf{e}_2 + y_2(0)\mathbf{e}_2\cdot\mathbf{e}_2 = y_2(0).\end{aligned}$$

Thus we may write the final solution for $\mathbf{x}$ as

$$\mathbf{x}(t) = (\mathbf{x}(0)\cdot\mathbf{e}_1)\,\mathbf{e}_1 e^{3t} + (\mathbf{x}(0)\cdot\mathbf{e}_2)\,\mathbf{e}_2 e^t. \qquad \square$$

It is instructive to compare this to the general result (true for any vector $\mathbf{x}$ in $\mathbf{R}^2$), whose demonstration we leave as an exercise:

$$\mathbf{x} = (\mathbf{x}\cdot\mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{x}\cdot\mathbf{e}_2)\,\mathbf{e}_2. \tag{2}$$

The general theory of systems of ordinary differential equations will not be needed in the rest of the course. The point of the above is to give a concrete example of the method of breaking multidimensional initial data into components which evolve in a simple fashion, and then writing the solution to the original problem as the sum of these evolved parts. Thus we decomposed $\mathbf{x}(0)$ according to (2), evolved each piece separately (this only required multiplying by e^{3t} and e^t, respectively), and then summed the results.

Derivation of the heat equation. Pages 99–100 of the textbook give a nice derivation of the heat equation which we followed quite closely in class. The derivation in the textbook is actually a bit more general since it allowed for heat sources located within the body. (If our sphere really were a cow, for example, these could represent heat due to metabolisation of food or muscle contractions.) Here we shall only use the so-called homogeneous heat equation, meaning the heat equation without sources, which we write as (we want to use k for something else in a moment)

$$\frac{\partial u}{\partial t} = D\nabla^2 u.$$

Examples of solutions to the one-dimensional heat equation without sources[1]. In this case we seek a function $u(x,t)$ which satisfies the equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}. \tag{3}$$

[1]The point of the first example is to motivate the introduction of separated solutions, while that of the second is to motivate the idea that a general solution can be written as a sum of separated ones.

We need to know something more than just this equation if we wish to determine u throughout all of space and time. By analogy with the system of ordinary differential equations above, we try specifying initial datum[2] $u(x, 0) = \sin kx$, for some constant[3]. Now at $t = 0$, the equation (3) will then become

$$\partial_t u(x,0) = D\partial_x^2 (\sin kx) = -k^2 D \sin kx = -k^2 D u(x,0). \tag{4}$$

This by itself does not really tell us much. However, it leads us to guess that we might be able to find a solution to the original heat equation by requiring (4) to hold for all t, not just $t = 0$. This gives the equation

$$\partial_t u(x,t) = -k^2 D u(x,t).$$

Now fix some $x = x_0$, and let u denote $u(x_0, t)$, so that the equation becomes the ordinary differential equation

$$\frac{du}{dt} = -k^2 D u.$$

By the theory of first-order linear equations, the solution to this will be $u = Ce^{-k^2 Dt}$, where C is some constant which can be determined by evaluating both sides at $t = 0$:

$$C = Ce^{-k^2 D \cdot 0} = u(0) = u(x_0, 0) = \sin kx_0.$$

Thus we obtain, for all x_0, $u(x_0, t) = \sin kx_0 e^{-k^2 Dt}$, which gives the function

$$u(x,t) = \sin kx e^{-k^2 Dt}.$$

We must now check whether this is actually a solution to the heat equation. We have

$$\begin{aligned} \frac{\partial u}{\partial t} &= \sin kx \left(-k^2 D e^{-k^2 Dt}\right) \\ D\frac{\partial^2 u}{\partial x^2} &= D\left(-k^2 \sin kx\right) e^{-k^2 Dt}, \end{aligned}$$

which are easily seen to be equal. Thus the function $u(x,t) = \sin kx e^{-k^2 Dt}$ is a solution to the heat equation satisfying $u(x, 0) = \sin kx$, as desired. (Compare this to the solution to the system of ordinary differential equations with initial datum $\mathbf{x}(0) = \mathbf{e}_1$.)

Let us now consider the initial datum $u = \sin k_1 x + \sin k_2 x$, $k_1 \neq k_2$. Unfortunately the above approach does not work in this case, since

$$\partial_x^2 u = -k_1 \sin k_1 x - k_2 \sin k_2 x$$

is not simply a multiple of u. In order to treat this case, we note that the heat equation is *linear*, by which we mean that any linear combination of solutions is also a solution. To see this for the case of two solutions, suppose that u_1 and u_2 are solutions of the heat equation (with the same constant D), and that a_1 and a_2 are constants. Then we see that

$$\begin{aligned} \frac{\partial}{\partial t}\left(a_1 u_1 + a_2 u_2\right) &= a_1 \frac{\partial u_1}{\partial t} + a_2 \frac{\partial u_2}{\partial t} \\ &= a_1 \left(D\nabla^2 u_1\right) + a_2 \left(D\nabla^2 u_2\right) \\ &= D\nabla^2 \left(a_1 u_1 + a_2 u_2\right), \end{aligned}$$

[2]The word 'data' is actually a Latin plural (so please never make the too-clever-by-half mistake of writing datæ as though data were a Latin singular; the author encountered this once!). The singular is datum. One could argue whether giving $u(x, 0)$ is giving a singular or a plural quantity. We use the singular here because we shall want to talk about multiple distinct *data* below.

[3]Throughout this course, when we say 'constant' we mean a number which does not depend on any of the variables in the question; in other words, a quantity constant in both space and time.

since a_1 and a_2, being constants, can be brought through the differentiation signs. Since our initial datum is a sum of initial data both of which we know how to handle, this suggests that we do something similar to what worked in the case of the system of ordinary differential equations above and work out the solution for each of the initial data separately. More specifically, let u_1 be the solution to the heat equation determined above with $u_1(x, 0) = \sin k_1 x$, and u_2 be that with $u_2(x, 0) = \sin k_2 x$, so that

$$u_1(x,t) = \sin k_1 x e^{-k_1^2 Dt}$$
$$u_2(x,t) = \sin k_2 x e^{-k_2^2 Dt}.$$

Since these are both solutions, their sum $u_1(x,t) + u_2(x,t)$ will also be; moreover, at $t = 0$ it will agree with the initial datum given above. Thus the solution to the heat equation with initial datum $\sin k_1 x + \sin k_2 x$ is

$$u(x,t) = \sin k_1 x e^{-k_1^2 Dt} + \sin k_2 x e^{-k_2^2 Dt}.$$

Note the similarity to the solution to the system of ordinary differential equations above.

Commentary. It should be clear from the foregoing how to handle the case of an initial datum which is a sum of any finite number of sine functions. A review of our method shows that it also works for cosine functions; hence we now know how to find a solution to the heat equation with initial datum any finite sum of sine and cosine functions. By itself this is still not much use. However, it turns out that almost any function on a finite interval (and in particular, any continuous function on a closed interval) can be expressed as a series – an infinite sum – of sine and cosine functions. Thus, if we can find a way of expressing our initial datum as such a sum, we can apply the above method to determine the solution for all future times. The reason why sine and cosine functions (and, as we shall see later, Legendre polynomials and Bessel functions) are particularly useful is that they turn out to be orthogonal with respect to certain inner products (generalisations of the dot product we are familiar with in $\mathbf{R}^2$ and $\mathbf{R}^3$ to spaces of functions). Recalling our method for computing $y_1(0)$, $y_2(0)$ in the first example above, we see that this should allow us to compute the coefficients in the expansion of our initial datum as a series in sine and cosine functions using inner products. Thus we must first discuss what we mean by an inner product, and what kind of inner product we can put on a space of functions.

Inner Products. We are all familiar with the dot product in $\mathbf{R}^3$: if $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, $w = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, then

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

The dot product is useful for finding projections (this is basically how we used it in the first example above). In particular, we have

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{i})\mathbf{i} + (\mathbf{v} \cdot \mathbf{j})\mathbf{j} + (\mathbf{v} \cdot \mathbf{k})\mathbf{k}.$$

It would be helpful to be able to extend this formula to spaces of functions. Now a vector has only a finite number of components while a function has essentially infinitely many components (speaking very loosely); thus it seems reasonable that the sum over components which worked to give us the dot product of two vectors should become an integral when we work with functions. More specifically, consider the function space

$$X = \{f : [a,b] \to \mathbf{R} | f \text{ is integrable and bounded}\};$$

on X we may define an inner product

$$(f, g) = \int_a^b f(x)g(x)dx.$$

It turns out that this inner product has many of the same properties as the dot product, and in particular can be used to separate out the different sine and cosine components of a function under appropriate circumstances. We shall take this up on Tuesday.