Consider the ordinary differential equation

$$\frac{d^2u}{dt^2} = u.$$

This has solutions $u = \sinh t$, $\cosh t$. These are easily seen to be linearly independent and thus to form a basis for the solution set of the above equation (exercise). Thus the general solution to this equation can be written

$$u(t) = a\cosh t + b\sinh t.$$

Now suppose that we are given initial conditions $u(0) = u_0$, $u'(0) = u_1$. These give

$$\begin{aligned} u_0 &= u(0) = a\cosh 0 + b\sinh 0 = a \\ u_1 &= u'(0) = a\sinh 0 + b\cosh 0 = b. \end{aligned}$$

Thus this particular solution is

$$u(t) = u(0)\cosh t + u'(0)\sinh t.$$

Now it is a general fact (and one with which we shall become much better acquianted as this class goes on) that for evolution equations, such as the one here, the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, or the wave equation $\frac{\partial^u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, there is a collection of data, usually termed *Cauchy data*, pertaining to and completely determined by the solution, which, when given at one particular point in time, determine the solution uniquely for all future points in time. For the equation above, Cauchy data at $t = 0$ would be the pair $(u(0), u'(0))$; for the heat equation, it might be $u(0, x)$, for all x; for the wave equation, it might be the pair $(u(0, x), u_t(0, x))$ (where $u_t = \frac{\partial u}{\partial t}$). It is worth noting, though, that there is nothing special about the choice of $t = 0$: specifying Cauchy data at any point in time will allow us to find the solution at all future points. (The equation above and the wave equation can also be solved backwards in time; this is not always possible for the heat equation. But this does not matter for the present considerations.) Moreover, since the Cauchy data is completely determined by the solution, having given it at one initial point, we are able to determine it at all future points. Thus, in some sense, instead of thinking of the evolution of just the solution, we should really think of the evolution of the Cauchy data.

In the context of our original ordinary differential equation, this suggests that we should consider not just the solution u but actually the pair $(u(t), u'(t))$, and see how this pair evolves with time. Suppose, as above, that we are given that at $t = 0$, $(u(0), u'(0)) = (u_0, u_1)$. Then from the foregoing we have

$$\begin{aligned} u(t) &= u_0\cosh t + u_1\sinh t \\ u'(t) &= u_0\sinh t + u_1\cosh t \end{aligned}$$

or in other words

$$(u(t), u'(t)) = u_0(\cosh t, \sinh t) + u_1(\sinh t, \cosh t).$$

In particular, if $u_0 = 1$, $u_1 = 0$, then we get the solution

$$(u^1(t), {u^1}'(t)) = (\cosh t, \sinh t),$$

while if $u_0 = 0$, $u_1 = 1$, we get the solution

$$(u^2(t), {u^2}'(t)) = (\sinh t, \cosh t).$$

Thus the general solution can be understood in this way: we decompose the initial data as $u_0 \cdot (1, 0) + u_1 \cdot (0, 1)$, evolve each piece separately, and sum the results.

In general, when we treat partial differential equations (even, in a modified way, those which are not evolution equations, such as Laplace's equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \cdots = 0$), we shall follow a similar procedure: find a particularly convenient decomposition of the Cauchy (or, in general, boundary) data such that each individual piece propagages in a simple way, and then sum all of these propagated pieces to obtain the final solution.

We shall make this much more clear as we go on.