APM 346, Homework 9. Due Monday, July 22, at 8.00 AM EDT. To be marked completed/not completed.

1. Using the eigenfunctions and eigenvalues for the Laplacian on the cylinder $C = \{(\rho, \phi, z) | \rho < 1, 0 \le z \le 1\}$ derived in class, solve the following problem on C:

$$\nabla^2 u = z \begin{cases} 0, & \rho < \frac{1}{2} \\ \rho^3 \cos 3\phi, & \frac{1}{2} < \rho < 1 \end{cases}, \quad u|_{\partial C} = 0.$$

Let us denote the right-hand side of Poisson's equation above by f. Then expanding

$$f = \begin{cases} 0, & \rho < \frac{1}{2} \\ \rho^3 \cos 3\phi, & \frac{1}{2} < \rho < 1 \end{cases} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho) \sin n\pi z \left(a_{nmi} \cos m\phi + b_{nmi} \sin m\phi\right),$$

we see as usual that $b_{nmi} = 0$ for all n, m, and i, while $a_{nmi} = 0$ unless $m = 3$, and in that case (using the standard normalisation integrals for $J_3(\lambda_{3i}\rho)$ and $\sin n\pi z$ on $[0, 1]$)

$$\begin{aligned} a_{n3i} &= \frac{4}{J_4^2(\lambda_{3i})} \int_{\frac{1}{2}}^{1} \int_0^1 \rho^3 z J_3(\lambda_{3i}\rho) \sin n\pi z \, dz \, \rho d\rho = \frac{4}{J_4^2(\lambda_{3i})} \int_{\frac{1}{2}}^{1} \rho^4 J_3(\lambda_{3i}\rho) \, d\rho \int_0^1 z \sin n\pi z \, dz \\ &= \frac{4(-1)^{n+1}}{\lambda_{3i} J_4^2(\lambda_{3i}) n\pi} \left[J_4(\lambda_{3i}) - \frac{1}{16} J_4\left(\frac{1}{2}\lambda_{3i}\right) \right], \end{aligned}$$

whence

$$u = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{4(-1)^n}{\lambda_{3i} J_4^2(\lambda_{3i}) n\pi \left(\lambda_{3i}^2 + n^2\pi^2\right)} \left(J_4(\lambda_{3i}) - \frac{1}{16} J_4\left(\frac{1}{2}\lambda_{3i}\right) \right) J_3(\lambda_{3i}\rho) \sin n\pi z \cos 3\phi.$$

2. Using the eigenfunctions and eigenvalues for the Laplacian on the unit ball $B = \{(r, \theta, \phi) | r < 1\}$ derived in class, solve the following problem on B:

$$\nabla^2 u = 3 \sin^2\theta \cos 2\phi \begin{cases} r^2, & r < \frac{1}{2} \\ 0, & \frac{1}{2} < r < 1 \end{cases}, \quad u|_{\partial B} = 0.$$

Again, we expand the right-hand side:

$$\begin{cases} r^2, & r < \frac{1}{2} \\ 0, & \frac{1}{2} < r < 1 \end{cases} = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{i=1}^{\infty} j_\ell(\kappa_{\ell i} r) P_{\ell m}(\cos\theta) \left(a_{\ell mi} \cos m\phi + b_{\ell mi} \sin m\phi\right),$$

whence as before we have $b_{\ell mi} = 0$ for all ℓ, m, and i, while $a_{\ell mi} = 0$ unless $m = 2$. Now $P_{22}(\cos\theta) = 3\sin^2\theta$, so since $\{P_{\ell 2}(\cos\theta)\}_{\ell=2}^{\infty}$ is a complete orthogonal set on $[0, \pi]$, we must also have $a_{\ell 2i} = 0$ unless $\ell = 2$. Finally, denoting the above right-hand side by f,

$$\begin{aligned} a_{22i} &= \frac{\left(f, j_2(\kappa_{2i} r) P_{22}(\cos\theta) \cos 2\phi\right)}{\left(j_2(\kappa_{2i} r) P_{22}(\cos\theta) \cos 2\phi, j_2(\kappa_{2i} r) P_{22}(\cos\theta) \cos 2\phi\right)} \\ &= \frac{2}{j_3^2(\kappa_{2i})} \int_0^{\frac{1}{2}} r^4 j_2(\kappa_{2i} r) \, dr = \frac{2}{j_3^2(\kappa_{2i})} \int_0^{\frac{1}{2}} r^{\frac{7}{2}} \sqrt{\frac{\pi}{2}} J_{\frac{5}{2}}(\kappa_{2i} r) \, dr \\ &= \frac{2}{\kappa_{2i} j_3^2(\kappa_{2i})} \sqrt{\frac{\pi}{2}} \left(\frac{1}{\sqrt{128}} J_{\frac{7}{2}}\left(\frac{1}{2}\kappa_{2i}\right) \right) = \frac{1}{8\sqrt{\kappa_{2i}} j_3^2(\kappa_{2i})} j_3\left(\frac{1}{2}\kappa_{2i}\right), \end{aligned}$$

so (since the eigenvalue corresponding to $j_m(\kappa_{mi} r) P_{\ell m}(\cos\theta) \cos m\phi$ is simply $-\kappa_{mi}^2$)

$$u = \sum_{i=1}^{\infty} -\frac{1}{8 j_3^2(\kappa_{2i}) \kappa_{2i}^{\frac{5}{2}}} j_3\left(\frac{1}{2}\kappa_{2i}\right) j_2(\kappa_{2i}\rho) P_{22}(\cos\theta) \cos 2\phi.$$

3. Solve the following problem on the unit cube Q:

$$\nabla^2 u = 0, \quad u|_{x=0} = u|_{x=1} = u|_{y=0} = u|_{y=1} = 0, \quad u|_{z=0} = \sin \pi x \sin 2\pi y, \quad u|_{z=1} = 0.$$

One way of doing this (which is not really detailed enough to count as a full solution on a test!) is to note that the solution will be a linear combination of $\sin \pi x \sin 2\pi y \cosh \pi\sqrt{5}z$ and $\sin \pi x \sin 2\pi y \sinh \pi\sqrt{5}z$, after which a little thought shows that the solution is exactly

$$\sin \pi x \sin 2\pi y \left(\cosh \pi\sqrt{5}z - \coth \pi\sqrt{5} \sinh \pi\sqrt{5}z\right).$$

More systematically, we note that the solution can be written in the form

$$u = \sum_{\ell,m=1}^{\infty} \sin \ell\pi x \sin m\pi y \left(a_{\ell m} \cosh \pi\sqrt{\ell^2+m^2}z + b_{\ell m} \sinh \pi\sqrt{\ell^2+m^2}z\right);$$

then the boundary conditions give that for $(\ell, m) \neq (1, 2)$

$$a_{\ell m} = 0, \quad a_{\ell m} \cosh \pi\sqrt{\ell^2+m^2} + b_{\ell m} \sinh \pi\sqrt{\ell^2+m^2} = 0,$$

whence it is easily seen that $a_{\ell m} = b_{\ell m} = 0$ for all $(\ell, m) \neq (1, 2)$; further,

$$a_{12} = 1, \quad a_{12} \cosh \pi\sqrt{5} + b_{12} \sinh \pi\sqrt{5} = 0,$$

so $b_{12} = -\coth \pi\sqrt{5}$ and we obtain $u = \sin \pi x \sin 2\pi y \left(\cosh \pi\sqrt{5}z - \coth \pi\sqrt{5} \sinh \pi\sqrt{5}z\right)$, as claimed.

4. Recall the function χ defined in problem 1 of assignment 8:

$$\chi(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1 \end{cases}.$$

Let u_0 denote the solution to problem 3. Solve the following problem on the unit cube Q:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{\partial Q} = u_0|_{\partial Q}, \quad u|_{t=0} = \chi(x)\chi(y)\chi(z).$$

[Optional: compute the coefficients in the series for u for two choices of ℓ, m, and n, one small (say $\ell = m = n = 1$) and another large (say $\ell, m, n > 10$). Compare the ratio of these coefficients for $t = 0$ and $t = 10$.]

Does the function u have a limit as $t \to +\infty$?

By what we did in class, this reduces to solving the two problems

$$\nabla^2 U_1 = 0, \quad U_1|_{\partial Q} = u_0|_{\partial Q},$$
$$\frac{\partial u_2}{\partial t} = \nabla^2 u_2, \quad u_2|_{\partial Q} = 0, \quad u_2|_{t=0} = \chi(x)\chi(y)\chi(z) - U_1;$$

now since u_0 satisfies $\nabla^2 u_0 = 0$, the first problem gives clearly $U_1 = u_0$, whence we need only to satisfy the problem

$$\frac{\partial u_2}{\partial t} = \nabla^2 u_2, \quad u_2|_{\partial Q} = 0, \quad u_2|_{t=0} = \chi(x)\chi(y)\chi(z) - u_0.$$

From problem 1 of assignment 8, we have the expansion

$$\chi(x)\chi(y)\chi(z) = \sum_{\ell,m,n=1}^{\infty} \frac{8}{\pi^3 \ell m n} \left((-1)^{\ell+1} + \cos \frac{1}{2}\ell\pi\right) \left((-1)^{m+1} + \cos \frac{1}{2}m\pi\right) \left((-1)^{n+1} + \cos \frac{1}{2}n\pi\right)$$
$$\cdot \sin \ell\pi x \sin m\pi y \sin n\pi z.$$

Now it is necessary to expand u_0 in the basis $\{\sin \ell\pi x \sin m\pi y \sin n\pi z\}$; the only tricky part of this is the expansion in the z direction. For this we note the following integral:

$$\begin{aligned}\int_0^1 e^{az}\sin n\pi z\,dz &= -\frac{\cos n\pi z}{n\pi}e^{az}\Big|_0^1 + \frac{a}{n\pi}\int_0^1 \cos n\pi z e^{az}\,dz \\ &= \frac{1}{n\pi}\left[1-(-1)^n e^a\right] + \frac{a}{n\pi}\left[\frac{\sin n\pi z}{n\pi}e^{az}\Big|_0^1 - \frac{a}{n\pi}\int_0^1 \sin n\pi z e^{az}\,dz\right]\end{aligned}$$

whence

$$\int_0^1 e^{az}\sin n\pi z\,dz = \frac{\frac{1}{n\pi}\left[1-(-1)^n e^a\right]}{1+\frac{a^2}{n^2\pi^2}} = \frac{n\pi\left[1-(-1)^n e^a\right]}{n^2\pi^2+a^2}.$$

From this we see easily that (since $\cosh x = \frac{1}{2}\left(e^x+e^{-x}\right)$ and $\sinh x = \frac{1}{2}\left(e^x-e^{-x}\right)$)

$$\begin{aligned}\int_0^1 \cosh \pi\sqrt{5}z \sin n\pi z\,dz &= \frac{n\pi\left[1-(-1)^n \cosh \pi\sqrt{5}\right]}{5\pi^2+n^2\pi^2}, \\ \int_0^1 \sinh \pi\sqrt{5}z \sin n\pi z\,dz &= -\frac{n\pi(-1)^n \sinh \pi\sqrt{5}}{5\pi^2+n^2\pi^2},\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 \left(\cosh \pi\sqrt{5}z - \coth \pi\sqrt{5}\sinh \pi\sqrt{5}z\right)\sin n\pi z\,dz \\ &\qquad = \frac{n\pi}{5\pi^2+n^2\pi^2}\left(\left[1-(-1)^n\cosh \pi\sqrt{5}\right] + \coth \pi\sqrt{5}(-1)^n \sinh \pi\sqrt{5}\right) \\ &\qquad = \frac{n\pi}{5\pi^2+n^2\pi^2}.\end{aligned}$$

From this we see that

$$u_0 = \frac{2}{\pi}\sin \pi x \sin 2\pi y \sum_{n=1}^{\infty}\frac{n}{5+n^2}\sin n\pi z,$$

so that

$$\begin{aligned}\chi(x)\chi(y)\chi(z) - u_0 = &\sum_{\substack{\ell,m,n=1\\(\ell,m)\neq(1,2)}}^{\infty} \frac{8}{\pi^3 \ell m n}\left((-1)^{\ell+1}+\cos\frac{1}{2}\ell\pi\right)\left((-1)^{m+1}+\cos\frac{1}{2}m\pi\right)\left((-1)^{n+1}+\cos\frac{1}{2}n\pi\right) \\ &\qquad \cdot \sin \ell\pi x \sin m\pi y \sin n\pi z \\ &+ \sin \pi x \sin 2\pi y \sum_{n=1}^{\infty}\left[\frac{8}{\pi^3 n}\left((-1)^{n+1}+\cos\frac{1}{2}n\pi\right) - \frac{2n}{\pi\left(5+n^2\right)}\right]\sin n\pi z\end{aligned}$$

and thus, by our usual method,

$$\begin{aligned}u_2 = &\sum_{\substack{\ell,m,n=1\\(\ell,m)\neq(1,2)}}^{\infty} \frac{8}{\pi^3 \ell m n}\left((-1)^{\ell+1}+\cos\frac{1}{2}\ell\pi\right)\left((-1)^{m+1}+\cos\frac{1}{2}m\pi\right)\left((-1)^{n+1}+\cos\frac{1}{2}n\pi\right) \\ &\qquad \cdot \sin \ell\pi x \sin m\pi y \sin n\pi z e^{-\pi^2\left(\ell^2+m^2+n^2\right)t} \\ &+ \sin \pi x \sin 2\pi y \sum_{n=1}^{\infty}\left[\frac{8}{\pi^3 n}\left((-1)^{n+1}+\cos\frac{1}{2}n\pi\right) - \frac{2n}{\pi\left(5+n^2\right)}\right]\sin n\pi z e^{-\pi^2\left(5+n^2\right)t}\end{aligned}$$

and the solution to our original problem is

$$
\begin{aligned}
u = {} & \sin \pi x \sin 2\pi y \left(\cosh \pi\sqrt{5}z - \coth \pi\sqrt{5} \sinh \pi\sqrt{5}z\right) \\
& + \sum_{\substack{\ell,m,n=1 \\ (\ell,m)\neq(1,2)}}^{\infty} \frac{8}{\pi^3 \ell m n} \left((-1)^{\ell+1} + \cos\frac{1}{2}\ell\pi\right)\left((-1)^{m+1} + \cos\frac{1}{2}m\pi\right)\left((-1)^{n+1} + \cos\frac{1}{2}n\pi\right) \\
& \qquad \cdot \sin \ell\pi x \sin m\pi y \sin n\pi z e^{-\pi^2\left(\ell^2+m^2+n^2\right)t} \\
& + \sin \pi x \sin 2\pi y \sum_{n=1}^{\infty} \left[\frac{8}{\pi^3 n}\left((-1)^{n+1} + \cos\frac{1}{2}n\pi\right) - \frac{2n}{\pi\left(5+n^2\right)}\right] \sin n\pi z e^{-\pi^2\left(5+n^2\right)t}.
\end{aligned}
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Clearly, $u \to \sin \pi x \sin 2\pi y \left(\cosh \pi\sqrt{5}z - \coth \pi\sqrt{5} \sinh \pi\sqrt{5}z\right)$ as $t \to +\infty$. We leave the optional part of this exercise to the reader.