APM 346, Homework 8. Due Monday, July 15, at 6.00 AM EDT. To be marked completed/not completed.

Using our derivation of the eigenfunctions and eigenvalues of the Laplacian in class, solve the following problems.

1. Write out a series expansion for the solution to the following problem on $Q = \{(x, y, z)|0 \leq x, y, z \leq 1\}$:

$$\nabla^2 u = \chi(x)\chi(y)\chi(z), \quad u|_{\partial Q} = 0,$$

where ∂Q is the boundary of the cube Q and

$$\chi(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1 \end{cases}.$$

We note the following integral:

$$\begin{aligned}\int_0^1 \chi(x) \sin \ell\pi x \, dx = \int_{\frac{1}{2}}^1 \sin \ell\pi x \, dx &= -\frac{\cos \ell\pi x}{\ell\pi}\bigg|_{\frac{1}{2}}^1 \\ &= \frac{(-1)^{\ell+1}}{\ell\pi} + \frac{\cos \frac{1}{2}\ell\pi}{\ell\pi} \\ &= \begin{cases} \frac{1}{\ell\pi}, & \ell \text{ odd} \\ \frac{1}{\ell\pi}\left((-1)^{\frac{\ell}{2}} - 1\right), & \ell \text{ even} \end{cases}.\end{aligned}$$

There is no convenient way to simplify a triple sum over a product of three versions of this last quantity, so we shall use the second-to-last line instead in the formulæ below. Thus we write

$$\begin{aligned}&\int_0^1 \int_0^1 \int_0^1 \chi(x)\chi(y)\chi(z) \sin \ell\pi x \sin m\pi x \sin n\pi x \, dx \, dy \, dz \\ &\qquad = \frac{1}{\pi^3 \ell mn}\left((-1)^{\ell+1} + \cos \frac{1}{2}\ell\pi\right)\left((-1)^{m+1} + \cos \frac{1}{2}m\pi\right)\left((-1)^{n+1} + \cos \frac{1}{2}n\pi\right),\end{aligned}$$

whence we may write

$$\begin{aligned}\chi(x)\chi(y)\chi(z) = \sum_{\ell,m,n=1}^{\infty} \frac{8}{\pi^3 \ell mn}&\left((-1)^{\ell+1} + \cos \frac{1}{2}\ell\pi\right)\left((-1)^{m+1} + \cos \frac{1}{2}m\pi\right)\left((-1)^{n+1} + \cos \frac{1}{2}n\pi\right) \\ &\cdot \sin \ell\pi x \sin m\pi y \sin n\pi z.\end{aligned}$$

From our general technique, if we denote the coefficients in the above series by $\chi_\ell \chi_m \chi_n$, then the coefficients $u_{\ell mn}$ in the series expansion for u will be given by

$$u_{\ell mn} = -\frac{1}{\pi^2(\ell^2 + m^2 + n^2)}\chi_\ell \chi_m \chi_n,$$

whence the solution for u will be

$$\begin{aligned}u = \sum_{\ell,m,n=1}^{\infty} -\frac{8}{\pi^5 \ell mn(\ell^2 + m^2 + n^2)}&\Bigg[\left((-1)^{\ell+1} + \cos \frac{1}{2}\ell\pi\right)\left((-1)^{m+1} + \cos \frac{1}{2}m\pi\right)\left((-1)^{n+1} + \cos \frac{1}{2}n\pi\right) \\ &\cdot \sin \ell\pi x \sin m\pi y \sin n\pi z\Bigg].\end{aligned}$$

2. Write out a series expansion for the solution to the following problem on $Q \times (0, +\infty)$, where Q is as in problem 1:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{\partial Q} = 0, \quad u|_{t=0} = \sin \pi x \sin \pi y,$$

where we denote an arbitrary point in $Q \times (0, +\infty)$ by (x, y, z, t).

We proceed similarly to question 1 and first calculate the expansion coefficients for the nonhomogeneous boundary term $\sin \pi x \sin \pi y$. Since

$$\int_0^1 \sin \ell\pi x \sin \pi x \, dx = \begin{cases} \frac{1}{2}, & \ell = 1 \\ 0, & \ell \neq 1 \end{cases}$$

and

$$\int_0^1 \sin n\pi z \, dz = -\frac{1}{n\pi} \cos n\pi z \Big|_0^1 = \frac{1}{n\pi} \left(1 - (-1)^n\right),$$

we see that

$$\int_0^1 \int_0^1 \int_0^1 \sin \pi x \sin \pi y \sin \ell\pi x \sin m\pi y \sin n\pi z \, dx \, dy \, dz = \begin{cases} \frac{1}{4n\pi} \left(1 - (-1)^n\right), & \ell = m = 1 \\ 0, & \text{otherwise} \end{cases},$$

whence

$$\sin \pi x \sin \pi y = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin \pi x \sin \pi y \sin (2k+1)\pi z.$$

We note that the eigenvalue corresponding to the kth term in the above sum is $-\pi^2(2 + (2k+1)^2)$ (since the kth term corresponds to the (ℓ, m, n) term in the original sum with $\ell = m = 1$ and $n = 2k + 1$). Thus the solution to our original problem is simply

$$u = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin \pi x \sin \pi y \sin (2k+1)\pi z e^{-\pi^2(2+(2k+1)^2)t}.$$