APM 346 (Summer 2019), Homework 7 solutions.

APM 346, Homework 7. Due Monday, July 8, at 6.00 AM EDT. To be marked completed/not completed.

1. Solve on $\{(\rho, \phi, z) | \rho < 2,\ \ 0 \le z \le 3\}$:

$$\nabla^2 u = 0,\ \ u|_{\rho=2} = 0,\ \ u|_{z=0} = \rho \cos\phi,\ \ u|_{z=3} = \rho \sin\phi.$$

We have the general series expansion (as introduced in class on Thursday; this form for the expansion turns out to be much more convenient for this particular problem than the one we have been using)

$$u = \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m\left(\frac{1}{2}\lambda_{mi}\rho\right) \left(\alpha_{mi} \cos m\phi \cosh \frac{1}{2}\lambda_{mi} z + \beta_{mi} \cos m\phi \sinh \frac{1}{2}\lambda_{mi} z \right.$$
$$\left. + \gamma_{mi} \sin m\phi \cosh \frac{1}{2}\lambda_{mi} z + \delta_{mi} \sin m\phi \sinh \frac{1}{2}\lambda_{mi} z\right),$$

where as usual λ_{mi} denotes the ith positive zero of J_m. At $z = 0$, this gives

$$\sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m\left(\frac{1}{2}\lambda_{mi}\rho\right) (\alpha_{mi} \cos m\phi + \gamma_{mi} \sin m\phi) = \rho\cos\phi,$$

whence we see that $\gamma_{mi} = 0$ for all m, i, while $\alpha_{mi} = 0$ unless $m = 1$, and in this case

$$\sum_{i=1}^{\infty} \alpha_{1i} J_1\left(\frac{1}{2}\lambda_{1i}\rho\right) = \rho,$$

whence

$$\alpha_{1i} = \frac{2}{2^2 J_2^2(\lambda_{1i})} \int_0^2 \rho^2 J_1\left(\frac{1}{2}\lambda_{1i}\rho\right) d\rho = \frac{1}{2J_2^2(\lambda_{1i})} \left(\frac{2}{\lambda_{1i}}\right)^3 \int_0^{\lambda_{1i}} x^2 J_1(x)\, dx$$
$$= \frac{4}{\lambda_{1i}^3 J_2^2(\lambda_{1i})} \lambda_{1i}^2 J_2(\lambda_{1i}) = \frac{4}{\lambda_{1i} J_2(\lambda_{1i})}.$$

At $z = 3$ we have

$$\sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m\left(\frac{1}{2}\lambda_{mi}\rho\right) \left(\alpha_{mi} \cos m\phi \cosh \frac{3}{2}\lambda_{mi} + \beta_{mi} \cos m\phi \sinh \frac{3}{2}\lambda_{mi} + \delta_{mi} \sin m\phi \sinh \frac{3}{2}\lambda_{mi}\right) = \rho^2 \sin\phi,$$

whence we see that $\alpha_{mi} \cosh \frac{3}{2}\lambda_{mi} + \beta_{mi} \sinh \frac{3}{2}\lambda_{mi} = 0$ for all m, i, which gives $\beta_{mi} = 0$ for $m \neq 1$ and $\beta_{1i} = -\coth \frac{3}{2}\lambda_{1i} \frac{4}{\lambda_{1i} J_2^2(\lambda_{1i})}$; also $\delta_{mi} = 0$ for $m \neq 1$, while

$$\sum_{i=1}^{\infty} \delta_{1i} \sinh \frac{3}{2}\lambda_{1i} J_1\left(\frac{1}{2}\lambda_{1i}\rho\right) = \rho,$$

whence we see from the above calculation for α_{1i} that

$$\delta_{1i} = \frac{4}{\lambda_{1i} J_2(\lambda_{1i}) \sinh \frac{3}{2}\lambda_{1i}}.$$

Thus finally

$$u = \sum_{i=1}^{\infty} J_1\left(\frac{1}{2}\lambda_{1i}\rho\right) \left[\left(\cosh \frac{1}{2}\lambda_{1i} z - \coth \frac{3}{2}\lambda_{1i} \sinh \frac{1}{2}\lambda_{1i} z\right) \cos\phi + \frac{\sinh \frac{1}{2}\lambda_{1i} z}{\sinh \frac{3}{2}\lambda_{1i}} \sin\phi\right] \frac{4}{\lambda_{1i} J_2(\lambda_{1i})}.$$

[It is worth noting how the quantity in parentheses interpolates between $\cos\phi$ at $z=0$ and $\sin\phi$ at $z=3$: the coefficients of $\cos\phi$ and $\sin\phi$ are exactly those linear combinations of $\cosh\frac{1}{2}\lambda_{1i}z$ and $\sinh\frac{1}{2}\lambda_{1i}z$ which are 1 at $z=0$ and $z=3$ and 0 at $z=3$ and $z=0$, respectively. In both cases, the remaining coefficient is exactly that needed for the ρ part to come out to ρ.]

2. Solve on $\{(\rho,\phi,z)|\rho<1,\ \ 0\le z\le 1\}$:

$$\nabla^2 u = 0,\ \ u|_{z=0} = u|_{z=1} = 0,\ \ u|_{\rho=1} = \begin{cases} -\phi, & 0<\phi<\pi, \\ \phi, & \pi<\phi<2\pi \end{cases}.$$

In this case we have the general series expansion

$$u = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} I_m(n\pi\rho)\left(a_{mn}\cos m\phi + b_{mn}\sin m\rho\right)\sin n\pi z.$$

Thus

$$\begin{aligned} a_{0n}I_0(n\pi) &= \frac{1}{\pi}\int_0^1 \sin n\pi z\left[\int_0^{\pi} -\phi\, d\phi + \int_{\pi}^{2\pi}\phi\, d\phi\right]dz = \frac{1}{\pi}\left(-\frac{1}{n\pi}\cos n\pi z\Big|_0^1\right)\left(-\frac{pi^2}{2}+\frac{1}{2}\left(4\pi^2-\pi^2\right)\right) \\ &= \frac{1}{n}\left(1-(-1)^n\right), \end{aligned}$$

while $b_{0i}=0$ by definition and for $m>0$

$$\begin{aligned} a_{mn}I_m(n\pi) &= \frac{2}{\pi}\int_0^1\left[\int_0^{\pi} -\phi\cos m\phi\sin n\pi z\, d\phi + \int_{\pi}^{2\pi}\phi\cos m\phi\sin n\pi z\, d\phi\right]dz \\ &= \frac{2}{\pi}\int_0^1 \sin n\pi z\left[-\frac{\phi}{m}\sin m\phi - \frac{1}{m^2}\cos m\phi\Big|_0^{\pi} + \frac{\phi}{m}\sin m\phi + \frac{1}{m^2}\cos m\phi\Big|_{\pi}^{2\pi}\right]dz \\ &= \frac{2}{\pi}\int_0^1 \sin n\pi z\left[\frac{1}{m^2}\left(1-(-1)^m\right) + \frac{1}{m^2}\left(1-(-1)^m\right)\right]dz = \frac{4}{n\pi^2 m^2}\left(1-(-1)^m\right)\left(1-(-1)^n\right), \end{aligned}$$

and

$$\begin{aligned} b_{mn}I_m(n\pi) &= \frac{2}{\pi}\int_0^1\left[\int_0^{\pi} -\phi\sin m\phi\sin n\pi z\, d\phi + \int_0^{2\pi}\phi\cos m\phi\sin n\pi z\, d\phi\right]dz \\ &= \frac{2}{\pi}\int_0^1 \sin n\pi z\left[\frac{\phi}{m}\cos m\phi - \frac{1}{m^2}\sin m\phi\Big|_0^{\pi} - \frac{\phi}{m}\cos m\phi + \frac{1}{m^2\sin m\phi}\Big|_{\pi}^{2\pi}\right]dz \\ &= \frac{2}{n\pi^2}\left(1-(-1)^n\right)\left[\frac{\pi}{m}(-1)^m - \left(\frac{\pi}{m}\left(2-(-1)^m\right)\right)\right] = -\frac{4}{mn\pi}\left(1-(-1)^n\right)\left(1-1(-1)^m\right), \end{aligned}$$

so finally

$$\begin{aligned} u = &\sum_{n=1}^{\infty}\frac{1}{n}\left(1-(-1)^n\right)\frac{I_0(n\pi\rho)}{I_0(n\pi)}\sin n\pi z \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{I_m(n\pi\rho)}{I_m(n\pi)}\left(\frac{4}{nm\pi}\left(1-(-1)^n\right)\left(1-(-1)^m\right)\right)\left[\frac{2}{m\pi}\cos m\phi - \sin m\phi\right]\sin n\pi z. \end{aligned}$$

3. Solve on $\{(\rho,\phi,z)|\rho<1,\ \ 0\le z\le 1\}$:

$$\nabla^2 u = 0,\ \ u|_{z=0} = \rho^2\cos 2\phi,\ \ u|_{z=1} = \rho^2\sin 2\phi,\ \ u|_{\rho=1} = \begin{cases} -\phi, & 0<\phi<\pi, \\ \phi, & \pi<\phi<2\pi \end{cases}.$$

[Hint: This is basically just problems 1 and 2 combined.]

We decompose this problem as $u = u_1 + u_2$, where u_1 is the solution to problem 2 and u_2 satisfies on $\{(\rho, \phi, z) | \rho < 1, 0 \le z \le 1\}$

$$\nabla^2 u_2 = 0, \quad u_2|_{\rho=1} = 0, \quad u_2|_{z=0} = \rho^2 \cos 2\phi, \quad u_2|_{\mathbf{z}=1} = \rho^2 \sin 2\phi.$$

This has solution

$$u_2 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mi}\rho) \Big(\alpha_{mi} \cos m\phi \cosh \lambda_{mi} z + \beta_{mi} \cos m\phi \sinh \lambda_{mi} z + \gamma_{mi} \sin m\phi \cosh \lambda_{mi} z + \delta_{mi} \sin m\phi \sinh \lambda_{mi} z \Big).$$

Now at $z = 0$

$$u_2 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mi}\rho) (\alpha_{mi} \cos m\phi + \gamma_{mi} \sin m\phi) = \rho^2 \cos 2\phi,$$

so just as in problem 1 we see that $\gamma_{mi} = 0$ for all m and all i, while $\alpha_{mi} = 0$ for $m \neq 2$ and we have after a standard calculation [we have done this calculation a number of times by this point, but on a test I would expect you to write it out anyway!]

$$\alpha_{2i} = \frac{2}{J_3^2(\lambda_{2i})} \int_0^1 \rho^3 J_2(\lambda_{2i}) \, d\rho = \frac{2}{\lambda_{2i} J_3(\lambda_{2i})}.$$

Similarly, at $z = 1$

$$u_2 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mi}\rho) (\alpha_{mi} \cos m\phi \cosh \lambda_{mi} + \beta_{mi} \cos m\phi \sinh \lambda_{mi} + \delta_{mi} \sin m\phi \sinh \lambda_{mi}) = \rho^2 \sin 2\phi,$$

so $\alpha_{mi} \cosh \lambda_{mi} + \beta_{mi} \sinh \lambda_{mi} = 0$ for all m and all i, meaning that $\beta_{mi} = 0$ for $m \neq 2$ while

$$\beta_{2i} = -\coth \lambda_{mi} \frac{2}{\lambda_{2i} J_3(\lambda_{2i})},$$

and $\delta_{mi} = 0$ for $m \neq 2$ while

$$\delta_{2i} \sinh \lambda_{2i} = \frac{2}{J_3^2(\lambda_{2i})} \int_0^1 \rho^3 J_2(\lambda_{2i}) \, d\rho = \frac{2}{\lambda_{2i} J_3(\lambda_{2i})},$$

giving

$$\delta_{2i} = \frac{2}{\lambda_{2i} J_3(\lambda_{2i}) \sinh \lambda_{2i}},$$

so

$$u_2 = \sum_{i=1}^{\infty} J_2(\lambda_{2i}\rho) \left(\frac{2}{\lambda_{2i} J_3(\lambda_{2i})} \right) \left[(\cosh \lambda_{2i} z - \coth \lambda_{2i} \sinh \lambda_{2i} z) \cos 2\phi + \frac{\sinh \lambda_{2i} z}{\sinh \lambda_{2i}} \sin 2\phi \right]$$

[note again how the coefficients on $\cos 2\phi$ and $\sin 2\phi$ interpolate between 0 and 1, exactly as with the solution in problem 1!] and finally

$$\begin{aligned} u = u_1 + u_2 = &\sum_{i=1}^{\infty} J_2(\lambda_{2i}\rho) \left(\frac{2}{\lambda_{2i} J_3(\lambda_{2i})} \right) \left[(\cosh \lambda_{2i} z - \coth \lambda_{2i} \sinh \lambda_{2i} z) \cos 2\phi + \frac{\sinh \lambda_{2i} z}{\sinh \lambda_{2i}} \sin 2\phi \right] \\ &+ \sum_{n=1}^{\infty} \frac{1}{n} (1 - (-1)^n) \frac{I_0(n\pi\rho)}{I_0(n\pi)} \sin n\pi z \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{I_m(n\pi\rho)}{I_m(n\pi)} \left(\frac{4}{nm\pi} (1 - (-1)^n)(1 - (-1)^m) \right) \left[\frac{2}{m\pi} \cos m\phi - \sin m\phi \right] \sin n\pi z. \end{aligned}$$

4. [Optional. This problem requires knowledge of basic complex function theory. I am only putting it here because I think it is exceptionally cool and can't resist.] Solve on $\{(\rho, \phi, z)|\rho < 1,\ \ 0 \le z \le 1\}$:

$$\nabla^2 u = 0,\ \ u|_{\rho=1} = 0,\ \ u|_{z=0} = 0,\ \ u|_{z=1} = \cos(\rho\cos\phi)\cosh(\rho\sin\phi).$$

[Hint: can you recognise the boundary datum at $z = 1$ as the real part of an analytic function of $x + iy$? Try writing out the power series of that function and solving the above problem term-by-term in that power series, noting that $x + iy = \rho e^{i\phi}$.]

[Sketch.] We note that $x + iy = \rho\cos\phi + i\rho\sin\phi = \rho e^{i\phi}$, so

$$\begin{aligned}\cos(x+iy) &= \cos x\cos iy - \sin x\sin iy = \cos x\cosh y + i\sin x\sinh y = \cos(\rho\cos\phi)\cosh(\rho\cos\phi) + \cdots \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x+iy)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\rho e^{i\phi}\right)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\rho^{2n}e^{2in\phi} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\rho^{2n}\left(\cos 2n\phi + i\sin 2n\phi\right),\end{aligned}$$

so

$$\cos(\rho\cos\phi)\cosh(\rho\sin\phi) = \Re\cos(x+iy) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\rho^{2n}\cos(2n+1)\phi.$$

Now the general solution to $\nabla^2 u = 0$ on the given region satisfying the first two boundary conditions is

$$u = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty}J_m(\lambda_{mi}\rho)\left(a_{mi}\cos m\phi + b_{mi}\sin m\phi\right)\sinh\lambda_{mi}z;$$

applying the boundary condition at $z = 1$ then gives

$$\sum_{m=0}^{\infty}\sum_{i=1}^{\infty}J_m(\lambda_{mi}\rho)\left(a_{mi}\cos m\phi + b_{mi}\sin m\phi\right)\sinh\lambda_{mi} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\rho^{2n}\cos 2n\phi.$$

From this we easily see that $b_{mi} = 0$ for all m, i, while $a_{mi} = 0$ when m is odd. If $m = 2n$ for some $n \in \mathbf{Z}$, $n \ge 0$, then we obtain

$$\begin{aligned}a_{2n,i}\sinh\lambda_{2n,i} &= \frac{(-1)^n}{(2n)!}\frac{2}{J_{2n+1}^2(\lambda_{2n,i})}\int_0^1\rho^{2n}J_{2n}(\lambda_{2n,i}\rho)\rho\,d\rho = \frac{(-1)^n 2}{(2n)!\lambda_{2n,i}J_{2n+1}(\lambda_{2n,i})}, \\ a_{2n,i} &= \frac{(-1)^n 2}{(2n)!\lambda_{2n,i}J_{2n+1}(\lambda_{2n,i})\sinh\lambda_{2n,i}},\end{aligned}$$

and

$$u = \sum_{n=0}^{\infty}\sum_{i=1}^{\infty}\frac{(-1)^n 2}{(2n)!\lambda_{2n,i}J_{2n+1}(\lambda_{2n,i})\sinh\lambda_{2n,i}}J_{2n}(\lambda_{2n,i}\rho)\cos 2n\phi\sinh\lambda_{2n,i}z.$$