APM 346, Homework 7. Due Monday, July 8, at 6.00 AM EDT. To be marked completed/not completed.

1. Solve on $\{(\rho,\phi,z)|\rho<2,\ \ 0\le z\le 3\}$:

$$\nabla^2 u=0,\ \ u|_{\rho=2}=0,\ \ u|_{z=0}=\rho\cos\phi,\ \ u|_{z=3}=\rho\sin\phi.$$

2. Solve on $\{(\rho,\phi,z)|\rho<1,\ \ 0\le z\le 1\}$:

$$\nabla^2 u=0,\ \ u|_{z=0}=u|_{z=1}=0,\ \ u|_{\rho=1}=\begin{cases}-\phi, & 0<\phi<\pi,\\ \phi, & \pi<\phi<2\pi\end{cases}.$$

3. Solve on $\{(\rho,\phi,z)|\rho<1,\ \ 0\le z\le 1\}$:

$$\nabla^2 u=0,\ \ u|_{z=0}=\rho^2\cos 2\phi,\ \ u|_{z=1}=\rho^2\sin 2\phi,\ \ u|_{\rho=1}=\begin{cases}-\phi, & 0<\phi<\pi,\\ \phi, & \pi<\phi<2\pi\end{cases}.$$

[Hint: This is basically just problems 1 and 2 combined.]

4. [Optional. This problem requires knowledge of basic complex function theory. I am only putting it here because I think it is exceptionally cool and can't resist.] Solve on $\{(\rho,\phi,z)|\rho<1,\ \ 0\le z\le 1\}$:

$$\nabla^2 u=0,\ \ u|_{\rho=1}=0,\ \ u|_{z=0}=0,\ \ u|_{z=1}=\cos\left(\rho\cos\phi\right)\cosh\left(\rho\sin\phi\right).$$

[Hint: can you recognise the boundary datum at $z=1$ as the real part of an analytic function of $x+iy$? Try writing out the power series of that function and solving the above problem term-by-term in that power series, noting that $x+iy=\rho e^{i\phi}$.]