APM 346 (Summer 2019), Homework 6 solutions.

APM 346, Homework 6. Due Wednesday, June 19, at 6.00 AM EDT. To be marked completed/not completed.

1. Solve the following boundary-value problem on the region $\{(\rho,\phi,z)|\rho<1, 0<z<1\}$ in cylindrical coordinates:

$$\nabla^2 u = 0, \qquad u|_{\rho=1} = 0, \qquad u|_{z=0} = 0, \qquad u|_{z=1} = 1.$$

We know from class that the general solution to Laplace's equation on the given region which satisfies the first boundary condition $u|_{\rho=1} = 0$ is of the form

$$u(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m\left(\lambda_{m,i}\rho\right)\left(a_{mi}\cos m\phi + b_{mi}\sin m\phi\right)\left(c_{mi}\cosh\lambda_{m,i}z + d_{mi}\sinh\lambda_{m,i}z\right),$$

where $\{\lambda_{m,i}\}_{i=1}^{\infty}$ is the set of all positive zeroes of $J_m(x)$. It is now just a matter of determining the coefficients in the above expansion which will make it satisfy the remaining boundary conditions. At $z=0$, we have

$$u_{z=0} = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m\left(\lambda_{m,i}\rho\right)\left(a_{mi}\cos m\phi + b_{mi}\sin m\phi\right)c_{mi} = 0;$$

thus (since $\{J_m\left(\lambda_{m,i}\rho\right)\cos m\phi, J_m\left(\lambda_{m,i}\rho\right)\sin m\phi\}$ is a complete orthogonal set on $[0,1]\times[0,2\pi]$) we must have $c_{mi}=0$ for all m and all i. Then we may absorb the coefficients d_{mi} into a_{mi} and b_{mi} and write

$$u(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m\left(\lambda_{m,i}\rho\right)\left(a_{mi}\cos m\phi + b_{mi}\sin m\phi\right)\sinh\lambda_{m,i}z.$$

At $z=1$, then, we have

$$u|_{z=1} = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m\left(\lambda_{m,i}\rho\right)\left(a_{mi}\cos m\phi + b_{mi}\sin m\phi\right)\sinh\lambda_{m,i} = 1.$$

Taking the inner product of this with functions $\cos m\phi$, $\sin m\phi$, $m>0$, we have

$$0 = (1,\cos m\phi) = \sum_{i=1}^{\infty} J_m\left(\lambda_{m,i}\rho\right)\left(a_{mi}\pi\right)\sinh\lambda_{m,i},$$
$$0 = (1,\sin m\phi) = \sum_{i=1}^{\infty} J_m\left(\lambda_{m,i}\rho\right)\left(b_{mi}\pi\right)\sinh\lambda_{m,i},$$

which gives (since $\{J_m\left(\lambda_{m,i}\rho\right)\}_{i=1}^{\infty}$ is a complete orthogonal set on $[0,1]$) that $a_{mi}=0$ and $b_{mi}=0$ for all $m>0$ and all i. Now $b_{0i}=0$ for all i by definition, so we are left simply with the condition

$$\sum_{i=1}^{\infty} a_{0i}J_0\left(\lambda_{0,i}\rho\right)\sinh\lambda_{0,i} = 1.$$

Using the orthogonality properties of the $J_0\left(\lambda_{0,i}\rho\right)$, we conclude that

$$a_{0i}\sinh\lambda_{0,i} = \frac{\left(1, J_0\left(\lambda_{0,i}\rho\right)\right)}{\left(J_0\left(\lambda_{0,i}\rho\right), J_0\left(\lambda_{0,i}\rho\right)\right)} = \frac{2}{J_1^2\left(\lambda_{0,i}\right)}\int_0^1 \rho J_0\left(\lambda_{0,i}\rho\right)\,d\rho.$$

Now

$$\int xJ_0(x)\,dx = xJ_1(x) + C,$$

so

$$\int_0^1 \rho J_0(\lambda_{0,i}\rho)\, d\rho = \frac{1}{\lambda_{0,i}^2}\int_0^{\lambda_{0,i}} xJ_0(x)\, dx = \frac{1}{\lambda_{0,i}^2}\lambda_{0,i}J_1(\lambda_{0,i}) = \frac{1}{\lambda_{0,i}}J_1(\lambda_{0,i}),$$

and

$$a_{0i} = \frac{2}{\lambda_{0,i}J_1(\lambda_{0,i})\sinh\lambda_{0,i}},$$

so finally

$$u = \sum_{i=1}^{\infty}\frac{2}{\lambda_{0,i}J_1(\lambda_{0,i})\sinh\lambda_{0,i}}J_0(\lambda_{0,i}\rho)\sinh\lambda_{0,i}z.$$

2. The same as 1, except with the condition $u|_{z=1} = 1$ replaced by $u|_{z=1} = \rho\cos\phi$.

The first few steps are of course the same as problem 1; thus we may start from the series expansion

$$u(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty}J_m(\lambda_{m,i}\rho)(a_{mi}\cos m\phi + b_{mi}\sin m\phi)\sinh\lambda_{m,i}z.$$

At $z = 1$ we now have

$$u|_{z=1} = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty}J_m(\lambda_{m,i}\rho)(a_{mi}\cos m\phi + b_{mi}\sin m\phi)\sinh\lambda_{m,i} = \rho\cos\phi,$$

whence as before we may conclude that $a_{mi} = 0$, $b_{mi} = 0$ for $m \neq 1$, all i, and also that $b_{1i} = 0$, while

$$\sum_{i=1}^{\infty}a_{1i}J_1(\lambda_{1,i}\rho)\sinh\lambda_{1,i} = \rho.$$

Thus we have as in 1

$$a_{1i}\sinh\lambda_{1,i} = \frac{(\rho, J_1(\lambda_{1,i}\rho))}{(J_1(\lambda_{1,i}\rho), J_1(\lambda_{1,i}\rho))} = \frac{2}{J_2^2(\lambda_{1,i})}\int_0^1 \rho^2 J_1(\lambda_{1,i}\rho)\, d\rho.$$

Now since

$$\int x^2 J_1(x)\, dx = x^2 J_2(x) + C,$$

we have

$$\int_0^1 \rho^2 J_1(\lambda_{1,i}\rho)\, d\rho = \frac{1}{\lambda_{1,i}^3}\lambda_{1,i}^2 J_2(\lambda_{1,i}) = \frac{1}{\lambda_{1,i}}J_2(\lambda_{1,i}),$$

so

$$a_{1i} = \frac{2}{\lambda_{1,i}J_2(\lambda_{1,i})\sinh\lambda_{1,i}}$$

and

$$u = \sum_{i=1}^{\infty}\frac{2}{\lambda_{1,i}J_2(\lambda_{1,i})\sinh\lambda_{1,i}}J_1(\lambda_{1,i}\rho)\cos\phi\sinh\lambda_{1,i}z.$$