APM 346 (Summer 2019), Homework 5 solutions.

APM 346, Homework 5. Due Monday, June 10, at 6.00 AM EDT. To be marked completed/not completed.

1. Solve the following boundary-value problem on the region $\{(r,\theta,\phi)|1<r<2\}$:

$$\nabla^2 u = 0, \qquad u|_{r=2} = \begin{cases} 1, & 0 \le \theta < \frac{\pi}{2} \\ -1, & \frac{\pi}{2} < \theta \le \pi \end{cases}, \qquad u_r|_{r=1} = \begin{cases} 0, & 0 \le \theta < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < \theta \le \pi \end{cases}.$$

[Hint: use Legendre polynomial identities to calculate $\int_0^1 P_\ell(x)\,dx$ and $\int_{-1}^0 P_\ell(x)\,dx$.]

Since the boundary conditions are azimuthally symmetric (and since we are solving on a spherical shell, which is an azimuthally symmetric region) we may write the general solution to Laplace's equation as

$$u = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(a_\ell r^\ell + b_\ell r^{-\ell-1}\right).$$

Now at $r=2$ we have

$$u|_{r=2} = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}}\right) = \begin{cases} 1, & 0 \le \theta < \frac{\pi}{2} \\ -1, & \frac{\pi}{2} < \theta \le \pi \end{cases}.$$

If we write this in terms of $x=\cos\theta$,it comes

$$u|_{r=2} = \sum_{\ell=0}^{\infty} P_\ell(x)\left(a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}}\right) = \begin{cases} -1, & -1 \le x < 0 \\ 1, & 0 < x \le 1 \end{cases};$$

thus the orthogonality and normalisation properties of the Legendre polynomials give

$$a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}} = \frac{2\ell+1}{2}\left[\int_{-1}^0 -P_\ell(x)\,dx + \int_0^1 P_\ell(x)\,dx\right] = \frac{2\ell+1}{2}\left[\int_0^1 [P_\ell(x) - P_\ell(-x)]\,dx\right];$$

if ℓ is even this will vanish, since P_ℓ will be an even function, while if ℓ is odd we have

$$a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}} = (2\ell+1)\int_0^1 P_\ell(x)\,dx.$$

Now we have the identity (from the lecture notes)

$$(2\ell+1)P_\ell(x) = P'_{\ell+1}(x) - P'_{\ell-1}(x);$$

thus for ℓ odd, say $\ell = 2k+1$,

$$a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}} = [P_{\ell+1}(1) - P_{\ell-1}(1) - (P_{\ell+1}(0) - P_{\ell-1}(0))] = -\left(P_{\ell+1}(0) - P_{\ell-1}(0)\right);$$

we shall show how to calculate this last expression shortly.

The second boundary condition (again working in terms of $x=\cos\theta$) gives

$$u_r|_{r=1} = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(\ell a_\ell - (\ell+1)b_\ell\right) = \begin{cases} 1, & -1 \le x < 0 \\ 0, & 0 < x \le 1 \end{cases},$$

so

$$\ell a_\ell - (\ell+1)b_\ell = \frac{2\ell+1}{2}\int_{-1}^0 P_\ell(x)\,dx.$$

If $\ell = 0$ this is just $\frac{1}{2}$, while if $\ell > 0$ is even it is $\frac{2\ell+1}{4}\int_{-1}^{1} P_\ell(x)\,dx = 0$. If ℓ is odd, say again $\ell = 2k+1$, we obtain

$$\ell a_\ell - (\ell+1)b_\ell = \frac{1}{2}\left[P_{\ell+1}(0) - P_{\ell-1}(0) - (P_{\ell+1}(-1) - P_{\ell-1}(-1))\right] = \frac{1}{2}\left(P_{\ell+1}(0) - P_{\ell-1}(0)\right),$$

since $P_{\ell+1}(-1) = (-1)^{\ell+1} = (-1)^{\ell-1} = P_{\ell-1}(-1)$. Thus for $\ell = 2k+1$ we have the system

$$\begin{aligned} a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}} &= -\left(P_{\ell+1}(0) - P_{\ell-1}(0)\right) \\ \ell a_\ell - (\ell+1)b_\ell &= \frac{1}{2}\left(P_{\ell+1}(0) - P_{\ell-1}(0)\right). \end{aligned}$$

This is a system of two linear equations in two unknowns and may be solved by a number of methods; perhaps the most systematic is to find the inverse of the coefficient matrix. We have the general formula (when the determinant $ad - bc$ is nonzero)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

which in the present case gives

$$\begin{pmatrix} 2^\ell & \frac{1}{2^{\ell+1}} \\ \ell & -(\ell+1) \end{pmatrix}^{-1} = \frac{1}{(\ell+1)2^\ell + \frac{\ell}{2^{\ell+1}}}\begin{pmatrix} \ell+1 & \frac{1}{2^{\ell+1}} \\ \ell & -2^\ell \end{pmatrix}.$$

Thus

$$\begin{aligned} \begin{pmatrix} a_\ell \\ b_\ell \end{pmatrix} &= \frac{1}{(\ell+1)2^\ell + \frac{\ell}{2^{\ell+1}}}\begin{pmatrix} \ell+1 & \frac{1}{2^{\ell+1}} \\ \ell & -2^\ell \end{pmatrix}\begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}\left(P_{\ell+1}(0) - P_{\ell-1}(0)\right) \\ &= \frac{2^{\ell+1}}{(\ell+1)2^{2\ell+2} + \ell}\begin{pmatrix} -(\ell+1) + \frac{1}{2^{\ell+2}} \\ -\ell - 2^{\ell-1} \end{pmatrix}\left(P_{\ell+1}(0) - P_{\ell-1}(0)\right). \end{aligned}$$

Now we have the general formula (this was discussed in lecture)

$$P_{2k}(0) = \frac{(-1)^k(2k-1)!!}{2^k k!};$$

thus

$$\begin{aligned} P_{2k+2}(0) - P_{2k}(0) &= \frac{(-1)^{k+1}(2k+1)!!}{2^{k+1}(k+1)!} - \frac{(-1)^k(2k-1)!!}{2^k k!} = \frac{(-1)^{k+1}(2k-1)!!}{2^k k!}\left(\frac{2k+1}{2k+2} + 1\right) \\ &= \frac{(-1)^{k+1}(2k-1)!!}{2^{k+1}(k+1)!}(4k+3), \end{aligned}$$

and since in the above formula we have $\ell = 2k+1$, we have finally the expression

$$\begin{pmatrix} a_{2k+1} \\ b_{2k+1} \end{pmatrix} = \frac{2^{2k+2}}{(2k+2)2^{4k+4} + 2k+1}\begin{pmatrix} -2k-2+\frac{1}{2^{2k+3}} \\ -2k-1-2^{2k} \end{pmatrix}\frac{(-1)^{k+1}(2k-1)!!}{2^{k+1}(k+1)!}(4k+3).$$

For ℓ even, $\ell > 0$, we have the system

$$\begin{aligned} a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}} &= 0 \\ \ell a_\ell - (\ell+1)b_\ell &= 0, \end{aligned}$$

which implies that $a_\ell = b_\ell = 0$ in this case; while for $\ell = 0$ we have instead the system

$$\begin{aligned} a_0 + \frac{b_0}{2} &= 0 \\ -b_0 &= \frac{1}{2}, \end{aligned}$$

which implies that $a_0 = \frac{1}{4}$, $b_0 = -\frac{1}{2}$. Thus finally we obtain the solution

$$u = \frac{1}{4} - \frac{1}{2r} + \sum_{k=0}^{\infty} P_{2k+1}(\cos\theta) \left[\frac{2^{k+1}(-1)^k(2k-1)!!(4k+3)}{[(k+1)2^{4k+4} + (2k+1)]\,(k+1)!} \right] \cdot \left[\left(2k+2 - \frac{1}{2^{2k+3}} \right) r^{2k+1} + \left(2k+1+2^{2k} \right) r^{-(2k+2)} \right].$$

2. Solve the following boundary-value problem on the region $\{(r,\theta,\phi) | r < 2\}$:

$$\nabla^2 u = 0, \qquad u|_{r=2} = x(1+y).$$

(Here $x = r\sin\theta\cos\phi$ and $y = r\sin\theta\sin\phi$ are the standard Cartesian coordinates corresponding to the given spherical coordinate system.)

This problem is much easier than problem 1. First of all, there is a straightforward way and a tricky way. We show the trick first. Since

$$\nabla^2 x(1+y) = 0$$

for all $x, y, z \in \mathbf{R}^3$, we see that $u = x(1+y)$ satisfies Laplace's equation on the given region; it also agrees with the boundary data, and thus it must be the solution we seek.

The straightforward way is rather longer (though also very instructive in our general technique) and goes as follows. Since we are solving on a region containing the origin, the general solution can be written as

$$u = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} P_{\ell,m}(\cos\theta) r^\ell \left(c_{\ell,m}\cos m\phi + d_{\ell,m}\sin m\phi \right).$$

Thus on the boundary $r = 2$ we have

$$\begin{aligned} u|_{r=2} &= \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} P_{\ell,m}(\cos\theta) 2^\ell \left(c_{\ell,m}\cos m\phi + d_{\ell,m}\sin m\phi \right) \\ &= x(1+y)|_{r=2} = 2\sin\theta\cos\phi + 4\sin^2\theta\cos\phi\sin\phi; \end{aligned}$$

since (see the lecture notes)

$$P_{1,1}(\cos\theta) = \sin\theta, \qquad P_{2,2}(\cos\theta) = 3\sin^2\theta,$$

we see that this last expression may be written as

$$2P_{1,1}(\cos\theta)\cos\phi + \frac{2}{3}P_{2,2}(\cos\theta)\sin 2\phi.$$

Orthogonality of the set $\{P_\ell\} \cup \{P_{\ell,m}\cos m\phi, P_{\ell,m}\sin m\phi\}$ then allows us to obtain

$$2^1 c_{1,1} = 2, \qquad 2^2 d_{2,2} = \frac{2}{3},$$

with all other $c_{\ell,m}$ and $d_{\ell,m}$ vanishing. This gives $c_{1,1} = 1$, $d_{1,1} = \frac{1}{6}$, and finally

$$\begin{aligned} u &= P_{1,1}(\cos\theta) r\cos\phi + \frac{1}{6}P_{2,2} r^2 \sin 2\phi \\ &= r\sin\theta\cos\phi + r^2\sin^2\theta\sin\phi\cos\phi = x + xy, \end{aligned}$$

the same as we obtained by the previous method.