APM 346 (Summer 2019), Homework 4 solutions.

APM 346, Homework 4. Due Monday, June 3, at 6.00 AM EDT. To be marked completed/not completed.

Consider the following boundary-value problem on $[0,1]\times[0,1]$:

$$\nabla^2 u = 0 \quad \text{on} \quad (0,1)\times(0,1), \qquad \begin{array}{cc} u(0,y)=0, & u_x(1,y)=-u(1,y), \\ u(x,0)=\sin n\pi x, & u(x,1)=\cos n\pi x, \end{array}$$

where $n \in \mathbf{Z}$, $n>0$ is some fixed positive integer.

1. Determine all separated solutions satisfying the homogeneous boundary conditions (these are the boundary conditions on $x=0$ and $x=1$ above).

We are looking for solutions of the form $u(x,y)=X(x)Y(y)$; substituting this in to Laplace's equation gives as usual

$$X''(x)Y(y)+X(x)Y''(y)=0,$$

whence dividing by u gives

$$\frac{X''}{X}+\frac{Y''}{Y}=0.$$

Since the first term depends only on x, and the second term only on y, the must individually be constant; since the data on the top and bottom edges of the boundary is oscillatory in x, we take X to be the oscillatory solution and Y to be the exponential one. Thus we have the equations

$$X''=-\lambda^2 X, \qquad Y''=\lambda^2 Y,$$

where we may take $\lambda>0$. The boundary conditions on the left and right sides of the boundary give

$$\begin{aligned} u(0,y) &= X(0)Y(y)=0, \\ \partial_x u(1,y) &= X'(1)Y(y) = -X(1)Y(y), \end{aligned}$$

and since we cannot have Y identically zero we must have $X(0)=0$, $X'(1)=-X(1)$. Thus X must satisfy the ordinary differential equation we have studied in Homework 2 and Homework 3; this means that we may take (we shall write the arbitrary constants in Y, as we did in the boundary-value problem we did in lecture) $X=\sin\lambda x$, where λ satisfies $\lambda=-\tan\lambda$. We must also have $Y=a_\lambda \sinh\lambda y+b_\lambda\cosh\lambda y$.

2. Assuming that the functions of x appearing in the separated solutions in 1 form a complete set on $[0,1]$, write out the general solution to $\nabla^2 u=0$ satisfying the first three boundary conditions above.

By Homework 3 we know that the set $\{\sin\lambda x | \lambda=-\tan\lambda\}$ is orthogonal on $[0,1]$, and we now assume (per the statement of the problem) that it is complete. This means that we can write the whole solution as a series in the separated solutions, i.e. (letting $\{\lambda_k\}_{k=1}^\infty$ denote the set of solutions to $\lambda=-\tan\lambda$ – which is easily seen to be discrete –)

$$u(x,y)=\sum_{k=1}^\infty (a_k\sinh\lambda_k y + b_k\cosh\lambda_k y)\sin\lambda_k x,$$

whence

$$\sin n\pi x = u(x,0) = \sum_{k=1}^\infty b_k \sin\lambda_k x,$$

and by our general results about expansions in complete orthogonal sets,

$$\begin{aligned} b_k &= \frac{\int_0^1 \sin n\pi x \sin\lambda_k x\,dx}{\int_0^1 \sin^2\lambda_k x\,dx} = \frac{\frac{1}{2}\left[\frac{1}{n\pi-\lambda_k}\sin\,(n\pi-\lambda_k)-\frac{1}{n\pi+\lambda_k}\sin\,(n\pi+\lambda_k)\right]}{\frac{1}{2}-\frac{\sin 2\lambda_k}{4\lambda_k}} \\ &= \frac{4n\pi(-1)^{n+1}\sin\lambda_k}{(n^2\pi^2-\lambda_k^2)\left(2-\frac{\sin 2\lambda_k}{\lambda_k}\right)}, \end{aligned}$$

so that the general solution satisfying the first three boundary conditions is

$$u(x,y)=\sum_{k=1}^{\infty}\left(a_k\sinh\lambda_k y+\frac{4n\pi(-1)^{n+1}\sin\lambda_k}{(n^2\pi^2-\lambda_k^2)\left(2-\frac{\sin 2\lambda_k}{\lambda_k}\right)}\cosh\lambda_k y\right)\sin\lambda_k x.$$

3. Finally, determine the unique solution to the full boundary-value problem.
Finally, we must have (writing b_k for the coefficient just given, for simplicity)

$$\cos n\pi x=u(x,1)=\sum_{k=1}^{\infty}(a_k\sinh\lambda_k+b_k\cosh\lambda_k)\sin\lambda_k x,$$

whence as in 2 we have, by our general results about expansions in complete sets of orthogonal functions,

$$\begin{aligned}
a_k\sinh\lambda_k+b_k\cosh\lambda_k &= \frac{4}{2-\frac{\sin 2\lambda_k}{\lambda_k}}\left(\int_0^1\cos n\pi x\sin\lambda_k x\,dx\right)\\
&=-\frac{2}{2-\frac{\sin 2\lambda_k}{\lambda_k}}\left(\frac{1}{n\pi+\lambda_k}\left(\cos\left(\lambda_k+n\pi\right)-1\right)+\frac{1}{\lambda_k-n\pi}\left(\cos\left(\lambda_k-n\pi\right)-1\right)\right)\\
&=-\frac{2}{2-\frac{\sin 2\lambda_k}{\lambda_k}}\left((-1)^n\cos\lambda_k-1\right)\frac{2\lambda_k}{\lambda_k^2-\pi^2n^2},
\end{aligned}$$

whence using the result from 2 and solving for a_k, we obtain

$$\begin{aligned}
a_k &= -\coth\lambda_k\left(\frac{4n\pi(-1)^{n+1}\sin\lambda_k}{(n^2\pi^2-\lambda_k^2)\left(2-\frac{\sin 2\lambda_k}{\lambda_k}\right)}\right)-\frac{2}{\left(2-\frac{\sin 2\lambda_k}{\lambda_k}\right)\sinh\lambda_k}\frac{2\lambda_k\left((-1)^n\cos\lambda_k-1\right)}{\lambda_k^2-\pi^2n^2}\\
&=\frac{4}{\left(2-\frac{\sin 2\lambda_k}{\lambda_k}\right)(n^2\pi^2-\lambda_k^2)\sinh\lambda_k}\left(n\pi(-1)^n\sin\lambda_k\cosh\lambda_k+\lambda_k\left((-1)^n\cos\lambda_k-1\right)\right),
\end{aligned}$$

and the final solution is given by the wonderful and marvelous expression

$$\begin{aligned}
u(x,y)=\sum_{k=1}^{\infty}\Bigg(&\frac{4\left(n\pi(-1)^n\sin\lambda_k\cosh\lambda_k+\lambda_k\left((-1)^n\cos\lambda_k-1\right)\right)}{\left(2-\frac{\sin 2\lambda_k}{\lambda_k}\right)(n^2\pi^2-\lambda_k^2)\sinh\lambda_k}\sinh\lambda_k y\\
&+\frac{4n\pi(-1)^{n+1}\sin\lambda_k}{(n^2\pi^2-\lambda_k^2)\left(2-\frac{\sin 2\lambda_k}{\lambda_k}\right)}\cosh\lambda_k y\Bigg)\sin\lambda_k x.
\end{aligned}$$

[Note. In 2 and 3, if you wish to use orthogonality of a certain set of functions, you must say how you know it is orthogonal (for example, by citing a specific result you have seen earlier in the course, or by giving a proof).
As noted above, orthogonality follows from Homework 3.]

The next two problems deal with Laplace's equation in spherical coordinates.
4. Consider the boundary-value problem on the region given by $\{(r,\theta,\phi)|1\le r\le 2\}$:

$$\nabla^2u=0,\quad 1<r<2,\quad u(r=1)=1,u_r(r=2)=-u(r=2).$$

Using our work with the Laplace equation in class, find the solution to this problem. [Hint: it depends only on r, not on θ or ϕ.]

Since the boundary data depends only on r, we posit a solution of the form $u = u(r)$; substituting this into our expression for Laplace's equation in spherical coordinates, we see that u must satisfy

$$u'' + \frac{2}{r}u' = 0,$$

and by our work with Laplace's equation we see that we are looking for a separated solution with $\ell = m = 0$, which means that it must be of the form $u = a + \frac{b}{r}$ for some constants a and b. (This can also be obtained directly from the above equation, without recourse to our more general work in class, of course.) The boundary conditions then give

$$\begin{aligned} u(1) &= a + b = 1, \\ u_r(2) &= -\frac{b}{r^2}\bigg|_{r=2} = -\frac{b}{4} = -u(2) = -a - \frac{b}{2}, \end{aligned}$$

whence we see that $b = -4a$, $a = -\frac{1}{3}$, $b = \frac{4}{3}$. Thus the solution is $u = -\frac{1}{3} + \frac{4}{3r}$.

5. Consider the same problem as in 4, but with the second boundary condition replaced by $u(r = 2) = \cos\theta$. Find the solution to this problem. [Hint: it can be written as a sum of two separated solutions.]

By the hint, we are looking for a solution which is a sum of two separated solutions. Now on the inner boundary we have simply $u = 1$, which does not depend on either θ or ϕ and thus (as in fact we saw in 4) looks like the kind of condition which can be fit with a solution having $m = \ell = 0$, while on the outer boundary we have $u = \cos\theta = P_1(\cos\theta)$, which can be fit with a solution having $m = 0$, $\ell = 1$. Thus we look for a solution of the form

$$u(r, \theta) = a_0 + \frac{b_0}{r} + \left(a_1 r + \frac{b_1}{r^2}\right)\cos\theta$$

(remember that the general separated solution to Laplace's equation with $m = 0$ is $\left(ar^\ell + \frac{b}{r^{\ell+1}}\right)P_\ell(\cos\theta)$). This expression satisfies Laplace's equation on the given region (note that the region does not contain the origin) by construction, so we need only determine the coefficients from the boundary conditions. On the inner boundary we have

$$\begin{aligned} u(1, \theta) &= a_0 + b_0 + (a_1 + b_1)\cos\theta \\ &= (a_0 + b_0)P_0(\cos\theta) + (a_1 + b_1)\,P_1(\cos\theta) = 1, \end{aligned}$$

whence we see that (since the $P_n(x)$ form an orthogonal set on $[-1, 1]$ and hence are linearly independent) we must have $a_0 + b_0 = 1$, $a_1 + b_1 = 0$. Similarly, on the outer boundary we have

$$\begin{aligned} u(2, \theta) &= a_0 + \frac{1}{2}b_0 + \left(2a_1 + \frac{1}{4}b_1\right)\cos\theta \\ &= \left(a_0 + \frac{1}{2}b_0\right)P_0(\cos\theta) + \left(2a_1 + \frac{1}{4}b_1\right)P_1(\cos\theta) = P_1(\cos\theta), \end{aligned}$$

whence we have by the same logic that $a_0 + \frac{1}{2}b_0 = 0$, $2a_1 + \frac{1}{4}b_1 = 1$. Putting all of these equations together, we see that we have $a_0 = -1$, $b_0 = 2$, $a_1 = \frac{4}{7}$, $b_1 = -\frac{4}{7}$, so that the full solution is

$$u(r, \theta) = -1 + \frac{2}{r} + \left(\frac{4}{7}r - \frac{4}{7r^2}\right)\cos\theta.$$

(Those of you who have studied electrodynamics may recognise the last term – $\frac{\cos\theta}{r^2}$ – as being the electrostatic potential of an electric dipole.)