APM 346 (Summer 2019), Homework 4.

APM 346, Homework 4. Due Monday, June 3, at 6.00 AM EDT. To be marked completed/not completed.

Consider the following boundary-value problem on $[0,1]\times[0,1]$:

$$\nabla^2 u = 0 \quad \text{on} \quad (0,1)\times(0,1), \qquad u(0,y)=0, \qquad u_x(1,y) = -u(1,y),$$
$$u(x,0) = \sin n\pi x, \qquad u(x,1) = \cos n\pi x,$$

where $n \in \mathbf{Z}$, $n > 0$ is some fixed positive integer.

1. Determine all separated solutions satisfying the homogeneous boundary conditions (these are the boundary conditions on $x = 0$ and $x = 1$ above).

2. Assuming that the functions of x appearing in the separated solutions in 1 form a complete set on $[0, 1]$, write out the general solution to $\nabla^2 u = 0$ satisfying the first three boundary conditions above.

3. Finally, determine the unique solution to the full boundary-value problem.

[NOTE. In 2 and 3, if you wish to use orthogonality of a certain set of functions, you must say how you know it is orthogonal (for example, by citing a specific result you have seen earlier in the course, or by giving a proof).]

The next two problems deal with Laplace's equation in spherical coordinates.

4. Consider the boundary-value problem on the region given by $\{(r,\theta,\phi) | 1 \le r \le 2\}$:

$$\nabla^2 u = 0, \quad 1 < r < 2, \quad u(r=1) = 1, u_r(r=2) = -u(r=2).$$

Using our work with the Laplace equation in class, find the solution to this problem. [Hint: it depends only on r, not on θ or ϕ.]

5. Consider the same problem as in 4, but with the second boundary condition replaced by $u(r = 2) = \cos\theta$. Find the solution to this problem. [Hint: it can be written as a sum of two separated solutions.]