APM 346, Homework 3. Due Monday, May 27, at 6.05 AM EDT. To be marked completed/not completed.

1. Recall the following boundary-value problem on the interval [0, 1] from Homework 2:

$$f'' = -\lambda^2 f, \qquad f(0) = 0, \qquad f'(1) = -f(1).$$

Show that if (λ_1, f_1) and (λ_2, f_2) are two solutions to this boundary-value problem, $\lambda_1, \lambda_2 > 0$, $\lambda_1 \neq \lambda_2$, then f_1 and f_2 are orthogonal with respect to the standard inner product $(f, g) = \int_0^1 f(x)\overline{g(x)}\,dx$. (You may use the solution posted on the course website, or work directly from the equation and boundary conditions above.)

There are two ways of doing this problem. First, we know that we may write (letting $i = 1, 2$)

$$f_i = a_i \sin \lambda_i x, \qquad \lambda_i = -\tan \lambda_i.$$

Thus

$$\begin{aligned}
(f_1, f_2) &= \int_0^1 f(x)\overline{f_2(x)}\,dx = \int_0^1 a_1\overline{a_2} \sin \lambda_1 x \sin \lambda_2 x\,dx = a_1\overline{a_2} \cdot \frac{1}{2} \in_0^1 \cos\left[(\lambda_1 - \lambda_2)x\right] - \cos\left[(\lambda_1 + \lambda_2)x\right]\,dx \\
&= \frac{1}{2} a_1\overline{a_2} \left[\frac{\sin\left[(\lambda_1 - \lambda_2)x\right]}{\lambda_1 - \lambda_2}\bigg|_0^1 - \frac{\sin\left[(\lambda_1 + \lambda_2)x\right]}{\lambda_1 + \lambda_2}\bigg|_0^1 \right] = \frac{1}{2} a_1\overline{a_2} \left[\frac{\sin(\lambda_1 - \lambda_2)}{\lambda_1 - \lambda_2} - \frac{\sin(\lambda_1 + \lambda_2)}{\lambda_1 + \lambda_2} \right] \\
&= \frac{1}{2} a_1\overline{a_2} \left[\frac{\sin\lambda_1 \cos\lambda_2 - \cos\lambda_1 \sin\lambda_2}{-\tan\lambda_1 + \tan\lambda_2} + \frac{\sin\lambda_1 \cos\lambda_2 + \cos\lambda_1 \sin\lambda_2}{\tan\lambda_1 + \tan\lambda_2} \right] \\
&= \frac{1}{2} a_1\overline{a_2} \left[\frac{\sin\lambda_1 \cos\lambda_2 - \cos\lambda_1 \sin\lambda_2}{(-\sin\lambda_1 \cos\lambda_2 + \cos\lambda_1 \sin\lambda_2)\frac{1}{\cos\lambda_1 \cos\lambda_2}} + \frac{\sin\lambda_1 \cos\lambda_2 + \cos\lambda_1 \sin\lambda_2}{(\sin\lambda_1 \cos\lambda_2 + \cos\lambda_1 \sin\lambda_2)\frac{1}{\cos\lambda_1 \cos\lambda_2}} \right] \\
&= \frac{1}{2} a_1\overline{a_2} \left[-\cos\lambda_1 \cos\lambda_2 + \cos\lambda_1 \cos\lambda_2 \right] = 0.
\end{aligned}$$

Alternatively, we may work directly from the equation. Since $\lambda_1 \neq \lambda_2$, at least one of $\lambda_1, \lambda_2 \neq 0$; we may assume that $\lambda_1 \neq 0$ without loss of generality (since our inner product satisfies $(f_1, f_2) = \overline{(f_2, f_1)}$). Then (note that we may assume that f_1 and f_2 are real, but this is not really necessary; we do assume however that λ is real, as we assumed in Homework 2)

$$\begin{aligned}
\int_0^1 f_1(x)\overline{f_2(x)}\,dx &= -\frac{1}{\lambda_1} \int_0^1 f_1''(x)\overline{f_2(x)}\,dx = -\frac{1}{\lambda_1} \left[f_1'(x)\overline{f_2(x)}\Big|_0^1 - \int_0^1 f_1'(x)\overline{f_2'(x)}\,dx \right] \\
&= -\frac{1}{\lambda_1} \left[f_1'(x)\overline{f_2(x)}\Big|_0^1 - \left[f_1(x)\overline{f_2'(x)}\Big|_0^1 - \int_0^1 f_1(x)\overline{f_2''(x)}\,dx \right] \right] \\
&= -\frac{1}{\lambda_1} \left[f_1'(x)\overline{f_2(x)}\Big|_0^1 - f_1(x)\overline{f_2'(x)}\Big|_0^1 - \lambda_2 \int_0^1 f_1(x)\overline{f_2(x)}\,dx \right],
\end{aligned}$$

whence we see that, solving for $\int_0^1 f_1(x)\overline{f_2(x)}\,dx$,

$$\left(1 - \frac{\lambda_2}{\lambda_1}\right) \int_0^1 f_1(x)\overline{f_2(x)}\,dx = -\frac{1}{\lambda_1} \left[-f_1(1)\overline{f_2(1)} - f_1(1)\left[-\overline{f_2(1)} \right] \right] = 0,$$

where we have used the boundary conditions. Since $\lambda_1 \neq \lambda_2$, this shows that $(f_1, f_2) = \int_0^1 f_1(x)\overline{f_2(x)}\,dx = 0$, as desired.

2. Solve the following boundary-value problem on $[0,1]\times[0,1]$:

$$\nabla^2 u = 0, \qquad f(x,0) = \begin{cases} 1, & x \in [0,\frac{1}{2}) \\ 0, & x \in (\frac{1}{2},1] \end{cases}, \qquad f(x,1) = \begin{cases} 0, & x \in [0,\frac{1}{2}) \\ 1, & x \in (\frac{1}{2},1] \end{cases},$$
$$f(0,y) = 0, \qquad f(1,y) = 0.$$

(You may use the expansion of $f(x,0)$ given in the lecture notes.)

[Erratum: please read 'u' for 'f' at each occurence in the foregoing. We apologise and hope this did not cause too much confusion.]

We begin by looking for separated solutions: suppose that $u(x,y) = X(x)Y(y)$; then we have

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0,$$

whence as discussed in lecture we must have $X'' = -\lambda^2 X$, $Y'' = \lambda^2 Y$, for some constant λ (which will be real since the boundary conditions force us to have $\frac{X''}{X} < 0$, and which we may then take to be positive[1]). These equations have solutions $X = a_\lambda \cos\lambda x + b_\lambda \sin\lambda x$, $Y = c_\lambda \cosh\lambda y + d_\lambda \sinh\lambda y$. Thus we posit that the full solution will have the form

$$u = \sum_\lambda \left(a_\lambda \cos\lambda x + b_\lambda \sin\lambda x\right)\left(c_\lambda \cosh\lambda y + d_\lambda \sinh\lambda y\right).$$

We may now apply the boundary conditions to determine λ and the coefficients in the above expansion. First of all, we apply the homogeneous conditions:

$$u(0,y) = \sum_\lambda a_\lambda \left(c_\lambda \cosh\lambda y + d_\lambda \sinh\lambda y\right) = 0$$

whence we take $a_\lambda = 0$;

$$u(1,y) = \sum_\lambda b_\lambda \sin\lambda \left(c_\lambda \cosh\lambda y + d_\lambda \sinh\lambda y\right) = 0,$$

whence we take $\lambda = n\pi$, $n \in \mathbf{Z}$, $n > 0$. Absorbing b_λ by writing

$$\alpha_n = b_{n\pi} c_{n\pi}, \quad \beta_n = b_{n\pi} d_{n\pi},$$

we may now write

$$u = \sum_{n=1}^{\infty} \sin n\pi x \left(\alpha_n \cosh n\pi y + \beta_n \sinh n\pi y\right).$$

We may now apply the other boundary conditions:

$$u(x,0) = \sum_{n=1}^{\infty} \sin n\pi x\,(\alpha_n) = \begin{cases} 1, & x \in [0,\frac{1}{2}) \\ 0, & x \in (\frac{1}{2},1] \end{cases}$$

We let $h(x)$ denote the function on the right-hand side above. Since, as discussed in lecture, the set $\{\sin n\pi x | n \in \mathbf{Z}, n > 0\}$ is complete on $[0,1]$, and since it is also orthogonal[2], we may calculate α_n as follows (exactly as was done in lecture):

$$\begin{aligned} \alpha_n &= \frac{(u, \sin n\pi x)}{\sin n\pi x, \sin n\pi x)} = \frac{\int_0^1 h(x) \sin n\pi x\ dx}{\int_0^1 \sin^2 n\pi x\,dx} = \frac{\int_0^{\frac{1}{2}} \sin n\pi x\,dx}{\int_0^1 \frac{1}{2}\left(1 - \cos 2n\pi x\right)\,dx} \\ &= \frac{-\frac{1}{n\pi}\cos n\pi x\big|_0^{\frac{1}{2}}}{\frac{1}{2}} = -\frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - 1\right]. \end{aligned}$$

[1] It should be noted that in principle $\lambda = 0$ should also be considered. However, it is readily seen that the solution for X in this case is of the form $ax + b$, which cannot satisfy the boundary conditions at $(0,y)$ and $(1,y)$ unless $a = b = 0$ and may thus be dropped.

[2] The instructor thinks he may have forgotten to demonstrate this point in class. It may be shewn easily as follows: $\int_0^1 \sin n\pi x \sin m\pi x = \frac{1}{2}\left[\frac{\sin[(n-m)\pi x]}{n-m}\Big|_0^1 - \frac{\sin[(n+m)\pi x]}{n+m}\Big|_0^1\right] = 0.$

Finally, the last boundary condition gives

$$u(x,1)=\sum_{n=1}^{\infty}\sin n\pi x\,(\alpha_n\cosh n\pi+\beta_n\sinh n\pi)=1-h,$$

whence we have

$$\begin{aligned}\alpha_n\cosh n\pi+\beta_n\sinh n\pi&=\frac{(1-h,\sin n\pi x)}{\sin n\pi x,\sin n\pi x)}=\frac{(1,\sin n\pi x)-(h,\sin n\pi x)}{(\sin n\pi x,\sin n\pi x)}\\&=2\int_0^1\sin n\pi x\,dx-\alpha_n=-\frac{2}{n\pi}\cos n\pi x|_0^1-\alpha_n\\&=-\frac{2}{n\pi}\left[(-1)^n-1\right]-\frac{2}{n\pi}\left[1-\cos\frac{n\pi}{2}\right]=-\frac{2}{n\pi}\left[(-1)^n-\cos\frac{n\pi}{2}\right],\end{aligned}$$

whence

$$\beta_n=-\alpha_n\coth n\pi-\frac{2}{n\pi\sinh n\pi}\left[(-1)^n-\cos\frac{n\pi}{2}\right]=-\frac{2}{n\pi\sinh n\pi}\left[\cosh n\pi\left(1-\cos\frac{n\pi}{2}\right)+(-1)^n-\cos\frac{n\pi}{2}\right].$$

Thus we have finally the grand expression[3]

$$u(x,y)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[\left(1-\cos\frac{n\pi}{2}\right)\cosh n\pi y+\frac{1}{\sinh n\pi}\left(\cosh n\pi\left(\cos\frac{n\pi}{2}-1\right)+\cos\frac{n\pi}{2}-(-1)^n\right)\sinh n\pi y\right]\cdot\sin n\pi x.$$

3. (a) Write x^4 on $(-1,1)$ as a series of Legendre polynomials. (Hint: the series has only finitely many terms. But you need to prove this!)

(b) (Optional) Is the series expansion from (a) valid outside of the interval $(-1,1)$? Is this likely to matter for our applications of Legendre polynomials?

(a) We have the first five Legendre polynomials (see p. 254 in the textbook)

$$P_0(x)=1,\quad P_1(x)=x,\quad P_2(x)=\frac{3}{2}x^2-\frac{1}{2},$$
$$P_3(x)=\frac{5}{2}x^3-\frac{3}{2}x,\quad P_4(x)=\frac{35}{8}x^4-\frac{15}{4}x^2+\frac{3}{8}.$$

Thus we may write $x^2=\frac{2}{3}\left(P_2+\frac{1}{2}P_0\right)$, whence

$$\begin{aligned}x^4&=\frac{8}{35}\left(P_4+\frac{15}{4}x^2-\frac{3}{8}P_0\right)=\frac{8}{35}\left(P_4+\frac{5}{2}\left(P_2+\frac{1}{2}P_0\right)-\frac{3}{8}P_0\right)\\&=\frac{8}{35}\left(P_4+\frac{5}{2}P_2+\frac{7}{8}P_0\right)=\frac{8}{35}P_4+\frac{4}{7}P_2+\frac{1}{5}P_0.\end{aligned}$$

[3] This is typical of the kinds of solutions one obtains by separation of variables. We should get some satisfaction out of our ability to construct such an expression! The author once read a biography of one Hugh Nibley ("A Consecrated Life", probably published by Deseret Book in 2002 or 2003, though the remaining bibliographical details escape me at the moment) in which he is reported to have written to his mother during training in meteorology (if my memory serves me correctly) in the US military prior to deployment in World War II, expressing the following sentiment: "We have become quite the little mathematician, and work great big problems sometimes passing within sight, almost, of the correct answer"! One of the author's colleagues at UC Berkeley expressed a similar sentiment regarding their common graduate quantum mechanics class, that she was learning how to actually solve quantum mechancis problems. For those of you who go on to study electrodynamics at the graduate level, the experience gained in producing solutions of this type will be extremely useful.

Alternatively, we may use the fact that the Legendre polynomials are orthogonal on the interval $[-1, 1]$ – since we have not yet discussed this we shall omit it for the moment. (The above calculation shows that the expansion can have only finitely many terms.)

(b) [NB This was added when it was anticipated that we would be able to discuss the orthogonality of the Legendre polynomials on $[-1, 1]$ before this homework was due. In that case, the point was that the expression in (a) would be derived using our general orthogonal function theory, in the which case it would not be clear a priori that it would hold outside of $[-1, 1]$. To prove that it does hold everywhere, though, it would be sufficient to note that polynomials equal on an interval are equal on the entire real line. This is not relevant for our applications of Legendre polynomials, though, since (as we shall see shortly) we are interested in Legendre polynomials of $\cos\theta$, and $\cos\theta \in [-1, 1]$ for all θ.]