APM 346, Homework 2. Due Monday, May 20, at 6 AM EDT. To be marked completed/not completed.

1. Use the identity $e^{3i\theta} = \left(e^{i\theta}\right)^3$ ($\theta \in \mathbf{R}$) to find an expression for $\cos 3\theta$ in terms of $\cos\theta$ and $\sin\theta$. We have

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= e^{3i\theta} = \left(e^{i\theta}\right)^3 = (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta).\end{aligned}$$

Since two complex numbers are equal if and only if their real and imaginary parts are equal, we see that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta.$$

2. Find all numbers $\lambda > 0$ for which there is a nonzero function f on $(0,1)$ satisfying

$$f'' = -\lambda^2 f, \qquad f(0) = 0, \qquad f'(1) = -f(1).$$

Also find the corresponding functions f. (Note: it is enough to find an equation which λ must satisfy. It is in general not possible to solve this equation.)

The general solution to the given differential equation is (using x as the independent variable) $f(x) = a\sin\lambda x + b\cos\lambda x$. The first boundary condition gives

$$f(0) = a\sin 0 + b\cos 0 = b = 0,$$

so that we may write $f(x) = a\sin\lambda x$. The second boundary condition then gives

$$f'(1) = a\lambda\cos\lambda = -f(1) = -a\sin\lambda.$$

Since we want $f \neq 0$ (note that this means that f and 0 are not the same function, i.e., that f is not identically zero; it does *not* mean that there is no x for which $f(x) = 0$!), we cannot have $a = 0$; thus we may cancel the a from this equation to obtain

$$\lambda = -\tan\lambda.$$

Thus, if $\lambda > 0$ is any solution to this equation, then $f(x) = a\sin\lambda x$ will satisfy the given boundary value problem for any a. (In principle, a could even be a complex number.)

3. (You need only do one of problems 3 and 4.) Suppose that $A_n \in \mathbf{R}$, $n = 0, 1, 2, \ldots$, $B_n \in \mathbf{R}$, $n = 1, 2, \ldots$, are such that

$$x = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n\cos 2n\pi x + B_n\sin 2n\pi x)$$

for $x \in (0,1)$. Find an expression for the A_n and B_n.

The set

$$\{1\} \cup \{\cos 2n\pi x,\ \sin 2n\pi x | n \in \mathbf{Z}, n > 0\}$$

is an orthogonal set, so we may calculate as follows, letting $(f,g) = \int_0^1 f(x)\overline{g(x)}\,dx$ denote the standard inner product on functions:

$$\begin{aligned}\frac{1}{2}A_0 &= \frac{(x,1)}{(1,1)} \\ &= \frac{\int_0^1 x\,dx}{\int_0^1 dx} = \frac{\frac{1}{2}x^2\big|_0^1}{1} = \frac{1}{2},\end{aligned}$$

so that $A_0 = 1$, while if $n > 0$

$$\begin{aligned}
A_n &= \frac{(x, \cos 2n\pi x)}{(\cos 2n\pi x, \cos 2n\pi x)} \\
&= \frac{\displaystyle\int_0^1 x \cos 2n\pi x \, dx}{\displaystyle\int_0^1 \cos^2 2n\pi x \, dx} = \frac{\left. x \frac{1}{2n\pi} \sin 2n\pi x \right|_0^1 - \displaystyle\int_0^1 \frac{1}{2n\pi} \sin 2n\pi x \, dx}{\displaystyle\int_0^1 \frac{1}{2} + \frac{1}{2} \cos 4n\pi x \, dx} \\
&= \frac{\left. \frac{1}{4n^2\pi^2} \cos 2n\pi x \right|_0^1}{\frac{1}{2}} = 0,
\end{aligned}$$

where we have used the fact that the integral of cosine over any integer number of periods is zero, and that $\cos 2n\pi = 1$, $\sin 2n\pi = 0$ for all integers n. Finally, we have

$$\begin{aligned}
B_n &= \frac{(x, \sin 2n\pi x)}{(\sin 2n\pi x, \sin 2n\pi x)} \\
&= \frac{\displaystyle\int_0^1 x \sin 2n\pi x \, dx}{\displaystyle\int_0^1 \sin^2 2n\pi x \, dx} = \frac{\left. -\frac{1}{2\pi n} x \cos 2\pi n x \right|_0^1 + \displaystyle\int_0^1 \frac{1}{2\pi n} \cos 2\pi n x \, dx}{\displaystyle\int_0^1 \frac{1}{2}(1 - \cos 4\pi n x) \, dx} \\
&= -\frac{1}{\pi n}.
\end{aligned}$$

4. (You need only do one of problems 3 and 4.) Suppose that $A_n \in \mathbf{C}$, $n = 0, 1, 2, \ldots$, are such that

$$x = \sum_{n=0}^{\infty} A_n e^{2in\pi x}$$

for $x \in (0, 1)$. Find an expression for the A_n.

Since $\{e^{2i\pi nx} | n \in \mathbf{Z}, n \geq 0\}$ is an orthonormal set, we may calculate as follows:

$$A_0 = (x, 1) = 1,$$

while for $n \neq 0$,

$$\begin{aligned}
A_n = (x, e^{2i\pi nx}) &= \int_0^1 x e^{-2i\pi nx} \, dx \\
&= \left. -\frac{1}{2i\pi n} x e^{-2i\pi nx} \right|_0^1 + \int_0^1 \frac{1}{2i\pi n} e^{-2i\pi nx} \, dx \\
&= -\frac{1}{2i\pi n} + \left. \frac{1}{4\pi^2 n^2} e^{-2i\pi nx} \right|_0^1 = -\frac{1}{2i\pi n},
\end{aligned}$$

where we have used $e^{2i\pi n} = 1$ for all integers n.

(Note. There was in fact a typographical error in the original problem, and the sum should have been extended from $-\infty$ to ∞; in other words, there is in fact no expansion of the form indicated in the problem statement. Technically, though, this does not affect our ability to solve the problem; and anyway the above calculation works for $n < 0$ just as well as for $n > 0$.)