APM346, Homework 1. Solutions.

1. Calculate the indicated derivatives.

(a) $\frac{d}{dx}\left(10x^6 - 5x^3 + 4x^2 - 7x + 1\right) = 60x^5 - 15x^2 + 8x - 7$.

(b) $\frac{d}{dx}\left(\ln\left[5x^2 - 3x + 100\right]\right) = \frac{10x-3}{5x^2-3x+100}$.

(c) $\frac{d}{dx}\left(e^{5x^{10}-10x^5+102}\right) = \left(50x^9 - 50x^4\right)e^{5x^{10}-10x^5+102}$.

(d) $\frac{d}{dx}(\sin 2x) = 2\cos 2x$.

(e) $\frac{d}{dx}(\cos kx) = -k\sin kx$, k a constant.

(f) $\frac{\partial}{\partial y}(\cos k_1 x\, \sin k_2 y) = k_2 \cos k_1 x \cos k_2 y$, k_1, k_2 constants.

(g)

$$\begin{aligned}
&\frac{\partial}{\partial z}\left(\sin^{-1}\left(\ln\left(\cos\left(\tan\left(xyz + x^2 + 10xy - 100\right)\right)\right)\right)\right) \\
&\quad = -\frac{1}{\sqrt{1 - \ln^2\left(\cos\left(\tan\left(xyz + x^2 + 10xy - 100\right)\right)\right)}} \frac{\sin\left(\tan\left(xyz + x^2 + 10xy - 100\right)\right)}{\cos\left(\tan\left(xyz + x^2 + 10xy - 100\right)\right)} \\
&\qquad \cdot xy\sec^2\left(xyz + x^2 + 10xy - 100\right) \\
&\quad = -\frac{1}{\sqrt{1 - \ln^2\left(\cos\left(\tan\left(xyz + x^2 + 10xy - 100\right)\right)\right)}} \tan\left(\tan\left(xyz + x^2 + 10xy - 100\right)\right) \\
&\qquad \cdot xy\sec^2\left(xyz + x^2 + 10xy - 100\right).
\end{aligned}$$

2. Evaluate the following expressions.

(a) $\nabla\left(x^2 + y^2\right) = 2x\mathbf{i} + 2y\mathbf{j}$.

(b) $\nabla\left(x^2 + y^2 - 2z^2\right) = 2x\mathbf{i} + 2y\mathbf{j} - 4z\mathbf{k}$.

(c) $\operatorname{div}(x\mathbf{i} + y\mathbf{j} + 10\mathbf{k}) = 1 + 1 + 0 = 2$.

(d) $\operatorname{div}\left(\nabla\left(x^2 + y^2 - 2z^2\right)\right) = \operatorname{div}(2x\mathbf{i} + 2y\mathbf{j} - 4z\mathbf{k}) = 0$.

(e) $\operatorname{div}(\nabla(e^y \sin x)) = \operatorname{div}(e^y \cos x\mathbf{i} + e^y \sin x\mathbf{j}) = -e^y \sin x + e^y \sin x = 0$.

3. Evaluate the following integrals. (You must show your work to get credit.)

(a) We use integration by parts:

$$\begin{aligned}
\int_0^{2\pi} x^2 \sin x\, dx &= -x^2 \cos x\Big|_0^{2\pi} + \int_0^{2\pi} 2x\cos x\, dx = -4\pi^2 + \left(2x\sin x\Big|_0^{2\pi} - \int_0^{2\pi} 2\sin x\, dx\right) \\
&= -4\pi^2 + 2\cos x\Big|_0^{2\pi} = -4\pi^2.
\end{aligned}$$

(b) We use integration by parts again:

$$\begin{aligned}
\int_0^{2\pi} x\sin(kx)\, dx &= -\frac{1}{k}x\cos(kx)\Big|_0^{2\pi} + \int_0^{2\pi} \frac{1}{k}\cos(kx)\, dx = -\frac{2\pi}{k}\cos(2\pi k) + \frac{1}{k^2}\sin(kx)\Big|_0^{2\pi} \\
&= -\frac{2\pi}{k}\cos(2\pi k) + \frac{1}{k^2}\sin(2\pi k).
\end{aligned}$$

(c) $\int_0^{+\infty} xe^{-x}\,dx = -xe^{-x}\big|_0^{+\infty} + \int_0^{+\infty} e^{-x}\,dx = -e^{-x}\big|_0^{+\infty} = 1$, where we can use L'Hôpital's rule to conclude $\lim_{x\to+\infty} xe^{-x} = 0$.

(d) This problem can be done two ways, one using a double integration by parts, and the other (for those who are comfortable working with complex functions) using complex exponentials. The first is as follows. We work with indefinite integrals:

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \cos x + \int e^x \sin x\,dx \\ &= e^x \cos x + \left(e^x \sin x - \int e^x \cos x\,dx\right)\end{aligned}$$

from which we easily see that $\int e^x \cos x\,dx = \frac{1}{2}e^x\left(\cos x + \sin x\right)$. From this it follows that

$$\int_0^{2\pi} e^x \cos x\,dx = \frac{1}{2}\left(e^{2\pi} - 1\right).$$

The other method is as follows:

$$\begin{aligned}\int e^x \cos x\,dx &= \int e^x \frac{e^{ix}+e^{-ix}}{2}\,dx = \frac{1}{2}\int e^{(1+i)x} + e^{(1-i)x}\,dx = \frac{1}{2}\left(\frac{e^{(1+i)x}}{1+i} + \frac{e^{(1-i)x}}{1-i}\right) \\ &= \frac{1}{4}\left((1-i)e^{(1+i)x} + (1+i)e^{(1-i)x}\right) = \frac{1}{4}2\mathrm{Re}\,(1-i)e^x\left(\cos x + i\sin x\right) \\ &= \frac{1}{2}e^x\left(\cos x + \sin x\right).\end{aligned}$$

From this the definite integral follows as before.

(e) $\int_0^{2\pi} \sin k_1 x \sin k_2 x\,dx$, $k_1, k_2 \in \mathbf{Z}$, $k_1 \neq k_2$.

This integral can be evaluated by using the trigonometric identity $\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$. In the present case, this gives

$$\begin{aligned}\int_0^{2\pi} \sin k_1 x \sin k_2 x\,dx &= \frac{1}{2}\int_0^{2\pi} \cos\left((k_1-k_2)x\right) - \cos\left((k_1+k_2)x\right)\,dx \\ &= \frac{1}{2}\left(\frac{\sin\left((k_1-k_2)x\right)}{k_1-k_2} - \frac{\sin\left((k_1+k_2)x\right)}{k_1+k_2}\right)\Bigg|_0^{2\pi} \\ &= 0,\end{aligned}$$

assuming $k_1 \neq -k_2$, $k_1, k_2 \neq 0$, and since $k_1 \neq k_2$. The case $k_1 = -k_2$ is essentially identical to $k_1 = k_2$ (since sin is odd) and is covered (up to a minus sign) by the solution to (f), and when either k_1 or k_2 is zero the integral is zero. (The author apologises for these oversights in setting the original problem; he should have written $k_1, k_2 \in \mathbf{Z}$, $k_1, k_2 > 0$.)

(f) Same as (e), but with $k_1 = k_2$.

Again, we assume $k_1 \neq 0$. In this case the identity above becomes $\sin^2 k_1 x = \frac{1}{2}\left(1 - \cos\left(2k_1 x\right)\right)$, and the above integral becomes

$$\int_0^{2\pi} \sin^2 k_1 x\,dx = \frac{1}{2}\int_0^{2\pi} 1 - \cos\left(2k_1 x\right)\,dx = \pi,$$

since the integral of cos will vanish as in (e).

(g) $\int_0^{2\pi} \sin k_1 x \cos k_2 x\,dx$, k_1, k_2 any two integers.

This is very similar to (e) but uses instead the identity $\sin a \cos b = \frac{1}{2}\left(\sin\left(a+b\right) + \sin\left(a-b\right)\right)$. The integral becomes, for $k_1 \neq \pm k_2$,

$$\begin{aligned}\int_0^{2\pi} \sin k_1 x \cos k_2 x &= \frac{1}{2}\int_0^{2\pi} \sin\left((k_1+k_2)x\right) + \sin\left((k_1-k_2)x\right)\, dx \\ &= -\frac{1}{2}\left(\frac{\cos\left((k_1+k_2)x\right)}{k_1+k_2} + \frac{\cos\left((k_1-k_2)x\right)}{k_1-k_2}\right)\Bigg|_0^{2\pi} \\ &= 0.\end{aligned}$$

If $k_1 = k_2$, then $\sin\left((k_1-k_2)x\right) = 0$ for all x, so its integral still vanishes, while if $k_1 = -k_2$ then the integral of $\sin\left((k_1+k_2)x\right)$ vanishes for the same reason. If $k_1 = k_2 = 0$ then the entire integrand vanishes. Thus the result above holds for all $k_1, k_2 \in \mathbf{Z}$.

4. Evaluate the following integrals.

(a) If $R = [0,\pi] \times [0,\pi]$, then

$$\begin{aligned}\iint_R \sin x\, \sin y\, dA &= \int_0^{\pi}\int_0^{\pi} \sin x\, \sin y\, dx\, dy \\ &= \int_0^{\pi} \sin y\, \left(-\cos x\right)\big|_0^{\pi}\, dy = 2\int_0^{\pi} \sin y\, dy = 4.\end{aligned}$$

(b) $\iint_R e^{-\left(x^2+y^2\right)}\, dA$, R the unit disk in the xy-plane.
In polar coordinates, R is represented by the set $\{(r,\theta) | r \leq 1\}$, and the integral becomes

$$\int_0^{2\pi}\int_0^1 e^{-r^2} r\, dr\, d\theta = \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_0^1\, d\theta = \pi\left(1 - e^{-1}\right).$$

(c) $\iiint_R \sin\left(x^2+y^2+z^2\right)^{\frac{3}{2}}\, dV$, R the unit ball in xyz-space.
In spherical polar coordinates, R is represented by the set $\{(r,\theta,\phi) | r \leq 1\}$, and the integral becomes

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\int_0^1 \sin r^3\, r^2 \sin\theta dr\, d\theta\, d\phi &= 2\pi\int_0^{\pi} \sin\theta\, \left(-\frac{1}{3}\cos r^3\right)\Big|_0^1\, d\theta \\ &= \frac{2\pi}{3}\left(1 - \cos 1\right)\, \left(-\cos\theta\right)\big|_0^{\pi} = \frac{4\pi}{3}\left(1 - \cos 1\right).\end{aligned}$$

5. Consider the two-dimensional vector space of functions on the interval $[0, 1]$

$$V = \{a \sin \pi x + b \cos \pi x | a, b \in \mathbf{R}\}.$$

Let $B = \{\sin \pi x, \cos \pi x\} \subset V$.

(a) Prove that B is a basis for V. (Hint: Wronskian!)
The Wronskian of the functions $\sin \pi x$ and $\cos \pi x$ is given by

$$W(x) = \begin{vmatrix} \sin \pi x & \cos \pi x \\ \pi \cos \pi x & -\pi \sin \pi x \end{vmatrix} = -\pi,$$

which is not zero on any interval. Thus the functions $\sin \pi x$ and $\cos \pi x$ are linearly independent on any open interval, and hence on the interval $[0, 1]$ itself, by the contrapositive of the proposition in the notes on the Wronskian on the course website. Since they span the space V by definition, they must then be a basis for V.

(b) Find the matrix representation $[T]_B$ of the operator T in the basis B, for (i) $T = \frac{d}{dx}$; (ii) $T = \frac{d^2}{dx^2}$.
(i) We evaluate T on the basis elements:

$$T(\sin \pi x) = \frac{d}{dx} \sin \pi x = \pi \cos \pi x, \qquad T(\cos \pi x) = -\pi \sin \pi x.$$

From this we see that (cf. the notes on linear algebra on the course website)

$$[T]_B = \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix}.$$

(ii) Again, we evaluate T on the basis elements:

$$T(\sin \pi x) = \frac{d}{dx} \pi \cos \pi x = -\pi^2 \sin \pi x, \qquad T(\cos \pi x) = -\pi^2 \cos \pi x.$$

From this we see that

$$[T]_B = \begin{pmatrix} -\pi^2 & 0 \\ 0 & -\pi^2 \end{pmatrix}.$$

We note that this is the square of the matrix in (i), as it should be.

6. Consider the differential equation $\frac{d^2 y}{dx^2} = -4y$.

(a) Find the set of all solutions to this equation.
Writing the equation as $\frac{d^2 y}{dx^2} + 4y = 0$, we have the characteristic equation $r^2 + 4 = 0$, which has the imaginary roots $r = \pm 2i$. This means that (as we could have determined by inspection in this case) the equation has solutions $\sin 2x$, $\cos 2x$; since (as we show in (b) in a moment) these are linearly independent, the solution set is $\{a \sin 2x + b \cos 2x | a, b \in \mathbf{R}\}$.

(b) Find a basis for this solution set. (You must prove that your answer is in fact a basis.)
We claim that $\{\sin 2x, \cos 2x\}$ is a basis for the solution set to this equation. We know from the theory of ordinary differential equations that the set of solutions to this equation is two-dimensional, so to show this it suffices to show that $\{\sin 2x, \cos 2x\}$ is linearly independent. This can be effected by computing its Wronskian:

$$W(x) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2 \cos 2x & -2 \sin 2x \end{vmatrix} = -2,$$

so as in 5(a) above this set is indeed linearly independent and hence (as noted in part (a) of this problem) a basis for the set of solutions to the equation.

(c) (Optional) Can you find the set of all solutions to $\frac{d^2y}{dx^2} + 4y = \sin 4x$?

By the theory of ordinary differential equations, the general solution to this equation will be the sum of a particular solution and the general solution to the corresponding homogeneous equation from (a), which we already know. Now we note that

$$\frac{d^2}{dx^2}\sin 4x = -16\sin 4x,$$

so that if $y = -\frac{1}{12}\sin 4x$,

$$\frac{d^2y}{dx^2} + 4y = -\frac{1}{12}\left(-16\sin 4x + 4\sin 4x\right) = \sin 4x,$$

and the set of all solutions to $\frac{d^2y}{dx^2} + 4y = \sin 4x$ is $\{-\frac{1}{12}\sin 4x + a\sin 2x + b\cos 2x | a, b \in \mathbf{R}\}$.

7. Find all (a) local and (b) global maxima of $f(x, y) = e^y \cos x$ on the rectangle $[0, 2\pi] \times [0, 1]$.

To find any local extrema, we compute the gradient and set it to zero:

$$\nabla e^y \cos x = -e^y \sin x\mathbf{i} + e^y \cos x\mathbf{j} = 0.$$

Since $e^y \neq 0$ for any y, this gives the system $\sin x = \cos x = 0$; but since $\sin^2 x + \cos^2 x = 1$, this is impossible. Thus this function has no local extrema in the rectangle (or anywhere in the plane, for that matter).

To find global extrema, we thus need only consider the function on the boundary. Now if $x = 0$ or $x = 2\pi$, we have $f(x, y) = e^y$, which (on $[0, 1]$) has a minimum of 1 at $y = 0$ and a maximum of e at $y = 1$. If $y = 0$ then $f(x, y) = \cos x$, which has a maximum of 1 at $x = 0$ and a minimum of -1 at $x = \pi$, while if $y = 1$ then $f(x, y) = e\cos x$, which has a maximum of e at $x = 0$ and a minimum of $-e$ at $x = \pi$. Putting all of this together, we see that the global maximum of $e^y \cos x$ is e, at the point $(0, 1)$, and the global minimum is $-e$, at $(\pi, 1)$. (Only the global maximum was required for this problem; the author put in the solution for the global minimum by mistake.)