APM 346, Homework 12. Due Monday, August 12, at 11.59 PM EDT. To be marked turned in/not turned in.

1. Using the eigenfunctions and eigenvalues for the Laplacian on the unit disk $D = \{(\rho, \phi) | \rho < 1\}$ derived in class, solve the following problem on $(0, +\infty) \times D$:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{\partial D} = 0, \quad u|_{t=0} = \rho^4 \cos 4\phi.$$

We have the eigenfunctions and eigenvalues (this is the set appropriate to the (spatial) geometry – the problem is given on D – as well as to the boundary conditions satisfied by u, which are homogeneous Dirichlet)

$$\mathbf{e}_{mi,\pm} = \begin{cases} J_m(\lambda_{mi}\rho)\cos m\phi \\ J_m(\lambda_{mi}\rho)\sin m\phi \end{cases}, \quad \lambda_{mi,\pm} = -\lambda_{mi}^2,$$

$m, i \in \mathbf{Z}$, $m \geq 0$, $i \geq 1$, where as usual λ_{mi} is the ith zero of $J_m(x)$. Now we may expand

$$u(t, \rho, \phi) = \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)(a_{mi}(t)\cos m\phi + b_{mi}(t)\sin m\phi); \tag{1}$$

substituting this into the equation $\frac{\partial u}{\partial t} = \nabla^2 u$ gives (since the eigenfunctions satisfy by definition $\nabla^2 \mathbf{e}_{mi,\pm} = \lambda_{mi,\pm}\mathbf{e}_{mi,\pm}$)

$$\begin{aligned}\frac{\partial}{\partial t}u &= \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)(a'_{mi}(t)\cos m\phi + b'_{mi}(t)\sin m\phi) \\ &= \nabla^2 u = \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} -\lambda_{mi}^2 J_m(\lambda_{mi}\rho)(a_{mi}(t)\cos m\phi + b_{mi}(t)\sin m\phi),\end{aligned}$$

whence, equating coefficients, we obtain

$$a'_{mi}(t) = -\lambda_{mi}^2 a_{mi}(t), \quad b'_{mi}(t) = -\lambda_{mi}^2 b_{mi}(t),$$

which have solution

$$a_{mi}(t) = a_{mi}(0)e^{-\lambda_{mi}^2 t}, \quad b_{mi}(t) = b_{mi}(0)e^{-\lambda_{mi}^2 t}. \tag{2}$$

Now substituting the expansion (1) into the initial conditions gives

$$\begin{aligned}u|_{t=0} &= \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)(a_{mi}(0)\cos m\phi + b_{mi}(0)\sin m\phi) \\ &= \rho^4 \cos 4\phi;\end{aligned}$$

comparing the ϕ dependence of the two sides (essentially, taking the inner product of both sides with $\cos 4\phi$), we see that $b_{mi}(0) = 0$ for all m, i, while $a_{mi}(0) = 0$ unless $m = 4$, and in that case we have

$$\sum_{i=1}^{\infty} a_{4i}(0) J_4(\lambda_{4i}\rho) \cos 4\phi = \rho^4 \cos 4\phi,$$

i.e.,

$$\sum_{i=1}^{\infty} a_{4i}(0) J_4(\lambda_{4i}\rho) = \rho^4.$$

This can be solved as usual:

$$\begin{aligned} a_{4i}(0) &= \frac{2}{J_5^2(\lambda_4 i)} \int_0^1 \rho^4 J_4(\lambda_{4i}\rho)\rho\, d\rho \\ &= \frac{2}{\lambda_{4i}^6 J_5^2(\lambda_4 i)} \int_0^{\lambda_{4i}} x^5 J_4(x)\, dx \\ &= \frac{2}{\lambda_{4i}^6 J_5^2(\lambda_{4i})}\, x^5 J_5(x)\Big|_0^{\lambda_{4i}} = \frac{2}{\lambda_{4i} J_5(\lambda_{4i})}. \end{aligned}$$

Substituting these results back into equations (2), we see that $b_{mi}(t) = 0$ for all t and all m and i, while $a_{mi}(t) = 0$ for all t unless $m = 4$, and in that case

$$a_{4i}(t) = \frac{2}{\lambda_{4i} J_5(\lambda_{4i})} e^{-\lambda_{4i}^2 t}.$$

Thus finally we have for u

$$u(t, \rho, \phi) = \sum_{i=1}^{\infty} \frac{2}{\lambda_{4i} J_5(\lambda_{4i})} J_4(\lambda_{4i}\rho) \cos 4\phi e^{-\lambda_{4i}^2 t}.$$

2. Using Fourier transforms in space, solve the following problem on $(0, +\infty) \times \mathbf{R}^1$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u|_{t=0} = e^{-x^2}, \quad u_t|_{t=0} = 0.$$

Fourier transforming in space gives

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -4\pi^2 k^2 \hat{u}, \quad \hat{u}|_{t=0} = \sqrt{\pi} e^{-\pi^2 k^2}, \quad \hat{u}_t|_{t=0} = 0.$$

The first equation has the general solution

$$\hat{u}(t, k) = a(k) \cos 2\pi kt + b(k) \sin 2\pi kt$$

(we may drop the absolute value signs around k since cos is even – so they don't matter – and sin is odd – which means we can drop them by redefining b(k) if necessary). Substituting this expression into the initial conditions gives

$$\begin{aligned} \hat{u}(0, k) &= a(k) = \sqrt{\pi} e^{-\pi^2 k^2}, \\ \hat{u}_t(0, k) &= -2\pi k a(k) \sin 2\pi k \cdot 0 + 2\pi k b(k) \cos 2\pi k \cdot 0 = 2\pi k b(k) = 0; \end{aligned}$$

thus $a(k) = \sqrt{\pi} e^{-\pi^2 k^2}$, while $b(k) = 0$ unless $k = 0$, and we may take $b(0) = 0$ for $k = 0$ without changing u (since $\sin 2\pi kt = 0$ for $k = 0$ anyway). Thus we have

$$\hat{u}(t, k) = \sqrt{\pi} e^{-\pi^2 k^2} \cos 2\pi kt = \frac{\sqrt{\pi}}{2} \left[e^{-\pi^2 k^2} e^{2\pi i k t} + e^{-\pi^2 k^2} e^{-2\pi i k t} \right].$$

We now wish to find u by taking the inverse Fourier transform of this expression. To do this, we recall the following property of Fourier transforms:

$$\mathcal{F}[f(\mathbf{x} - \boldsymbol{\alpha})](\mathbf{k}) = e^{-2\pi i \mathbf{k} \cdot \boldsymbol{\alpha}} \hat{f}(\mathbf{k}),$$

or in other words,

$$\mathcal{F}^{-1} \left[e^{-2\pi i \mathbf{k} \cdot \boldsymbol{\alpha}} \hat{f}(\mathbf{k}) \right] (\mathbf{x}) = f(\mathbf{x} - \boldsymbol{\alpha}).$$

This gives easily, since $\sqrt{\pi}e^{-\pi^2k^2} = \mathcal{F}[e^{-x^2}]$,

$$u(t,x) = \mathcal{F}^{-1}[\hat{u}(t,k)](x) = \frac{1}{2}\left[e^{-(x+t)^2} + e^{-(x-t)^2}\right],$$

which is the desired answer.

3. Again using Fourier transforms, consider the following problem on $(0,+\infty)\times\mathbf{R}^1$; give an integral expression for the solution, and evaluate it as far as possible:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u, \quad u|_{t=0} = e^{-x^2}.$$

What is the behaviour of this solution in the limit $t \to +\infty$?

In this case, Fourier transforming gives

$$\frac{\partial \hat{u}}{\partial t} = -4\pi^2k^2\hat{u} + \hat{u} = (1-4\pi^2k^2)\hat{u}, \quad \hat{u}|_{t=0} = \sqrt{\pi}e^{-\pi^2k^2}.$$

This problem clearly has the solution

$$\begin{aligned}\hat{u}(t,k) &= \hat{u}(0,k)e^{(1-4\pi^2k^2)t} = \sqrt{\pi}e^{-\pi^2k^2}e^{(1-4\pi^2k^2)t} \\ &= \sqrt{\pi}e^te^{-\pi^2k^2(1+4t)},\end{aligned}$$

whereupon taking an inverse Fourier transform gives [evidently this problem is easier than I intended; by 'an integral expression' I presumably meant a Fourier integral, but it seems that I thought that the integral might not be easy to evaluate, which is evidently not the case]

$$\begin{aligned}u(t,x) &= \sqrt{\pi}e^t\sqrt{\frac{\pi}{\pi^2(1+4t)}}e^{-\frac{x^2}{1+4t}} \\ &= \frac{1}{\sqrt{1+4t}}e^te^{-\frac{x^2}{1+4t}}\end{aligned}$$

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This expression clearly goes to infinity for all x in the limit as $t \to \infty$. More detailed information can however be obtained from the expression for the Fourier transform above: we see that if $4\pi^2k^2 < 1$, then $\hat{u}(t,k) \to \infty$ as $t \to \infty$, while if $4\pi^2k^2 > 1$, then $\hat{u}(t,k) \to 0$ as $t \to \infty$. Thus, essentially, while the solution does in fact go to infinity in the limit of long times, only small k (essentially, 'long-wavelength', i.e., large spatial scale) modes remain in this limit.