APM 346, Homework 11, solutions. Due Monday, August 5, at 6.00 AM EDT. To be marked completed/not completed.

1. Using the eigenfunctions derived in homework 10, problem 1, construct the Green's function on Q satisfying

$$\nabla^2_{\mathbf{x}} G(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}'), \quad \left.\frac{\partial G}{\partial n}\right|_{\mathbf{x}\in\partial Q} = 0,$$

and use it to find a series expansion for the solution to the following problem on Q:

$$\nabla^2 u = \sin 2\pi x \sin 2\pi y \sin 2\pi z, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial Q} = 1.$$

[The following question is worth considering: What would happen if we replaced $\sin 2\pi x \sin 2\pi y \sin 2\pi z$ by $\sin \pi x \sin \pi y \sin \pi z$ above?]

We have the eigenfunctions

$$\mathbf{e}_{\ell mn} = \cos \ell\pi x \cos m\pi y \cos n\pi z, \quad \ell, m, n \in \mathbf{Z}, \ell, m, n \geq 0,$$

with corresponding eigenvalues

$$\lambda_{\ell mn} = -\pi^2\left(\ell^2 + m^2 + n^2\right).$$

Formally, then, we have the Gren's function

$$G(\mathbf{x},\mathbf{x}') = -\sum_{\ell,m,n} \frac{8}{\lambda_{\ell mn}} \cos \ell\pi x \cos m\pi y \cos n\pi z \cos \ell\pi x' \cos m\pi y' \cos n\pi z'.$$

However, in this case we have a zero eigenvalue $\lambda_{000} = 0$, and the corresponding term in the above sum is undefined. We shall show at the end of this document that the ordinary formulas work just as well in this case, if we drop the terms with $\lambda_{\ell mn} = 0$ from the above sum and assume that $(f, \mathbf{e}_{\ell mn}) = 0$ for all such ℓmn. For the moment we show how to apply this to solve the current problem. We have the Green's function

$$G(\mathbf{x},\mathbf{x}') = \sum_{(\ell,m,n)\neq(0,0,0)} \frac{8}{\pi^2\left(\ell^2+m^2+n^2\right)} \cos \ell\pi x \cos m\pi y \cos n\pi z \cos \ell\pi x' \cos m\pi y' \cos n\pi z'.$$

Now applying the formula

$$u = -\int_Q G(\mathbf{x},\mathbf{x}')\nabla^2 u(\mathbf{x}')\, d\mathbf{x}' + \int_{\partial Q} G(\mathbf{x},\mathbf{x}')\frac{\partial u}{\partial n'} - u(\mathbf{x}')\frac{\partial G}{\partial n'}\, dS',$$

and using the fact that G satisfies homogeneous Neumann conditions on ∂Q, we have

$$u = -\int_Q G(\mathbf{x},\mathbf{x}')\sin 2\pi x' \sin 2\pi y' \sin 2\pi z'\, d\mathbf{x}' + \int_{\partial Q} G(\mathbf{x},\mathbf{x}')\, dS'.$$

Now we note that, for $\ell \neq 2$,

$$\begin{aligned}
\int_0^1 \cos \ell\pi x' \sin 2\pi x'\, dx' &= \frac{1}{2}\int_0^1 \sin\left[(\ell+2)\pi x'\right] - \sin\left[(\ell-2)\pi x'\right]\, dx' \\
&= -\frac{1}{2\pi}\left[\frac{1}{\ell+2}\cos\left[(\ell+2)\pi x'\right] - \frac{1}{\ell-2}\cos\left[(\ell-2)\pi x'\right]\right]\Bigg|_0^1 \\
&= \frac{1}{2\pi}\left[\frac{1}{\ell+2}(1-(-1)^\ell) - \frac{1}{\ell-2}(1-(-1)^\ell)\right] = -\frac{2}{\pi(\ell^2-4)}(1-(-1)^\ell),
\end{aligned}$$

while if $\ell = 2$ we have
$$\int_0^1 \cos 2\pi x' \sin 2\pi x'\, dx' = \frac{1}{2}\int_0^1 \sin 4\pi x'\, dx' = 0.$$
Thus the first integral above becomes
$$\begin{aligned}
&-\int_Q G(\mathbf{x},\mathbf{x}')\sin 2\pi x' \sin 2\pi y' \sin 2\pi z'\, d\mathbf{x}' \\
&\quad = \sum_{\substack{(\ell,m,n)\neq(0,0,0)\\ \ell,m,n\neq 2}} \frac{8}{\pi^2(\ell^2+m^2+n^2)}\left(-\frac{1}{\pi^3(\ell^2-4)(m^2-4)(n^2-4)}\right) \\
&\qquad \cdot 8\cdot(1-(-1)^\ell)(1-(-1)^m)(1-(-1)^n)\cos\ell\pi x\cos m\pi y\cos n\pi z.
\end{aligned}$$
For the second integral, we note that ∂Q is a union of six squares, namely
$$\begin{aligned}
&\{(x,y,z)|x\in\{0,1\},(y,z)\in[0,1]^2\} \\
&\qquad \cup\{(x,y,z)|y\in\{0,1\},(x,z)\in[0,1]^2\} \\
&\qquad\qquad \cup\{(x,y,z)|z\in\{0,1\},(x,y)\in[0,1]^2\};
\end{aligned}$$
these are the left and right, front and back, and top and bottom sides, respectively. Now the part of the second integral corresponding to the first of these would be
$$\begin{aligned}
&\int_0^1\int_0^1 G(\mathbf{x},\mathbf{x}')|_{x'=0}\, dy'\, dz' + \int_0^1\int_0^1 G(\mathbf{x},\mathbf{x}')|_{x'=1}\, dy'\, dz' \\
&\quad = 8\sum_{\substack{(\ell,m,n)\neq(0,0,0)\\ m,n\neq 2}} \frac{\cos\ell\pi x\cos m\pi y\cos n\pi z}{\pi^2(\ell^2+m^2+n^2)}(1+(-1)^\ell)\int_0^1\int_0^1 \cos m\pi y'\cos n\pi z'\, dy'\, dz' \\
&\quad = 8\sum_{\ell=1}^\infty \frac{1+(-1)^\ell}{\pi^2\ell^2}\cos\ell\pi x,
\end{aligned}$$
since the integral is zero unless $m = n = 0$ (and the final sum begins at $\ell = 1$ since we cannot have $\ell = m = n = 0$). Similar results would hold for the integrals over the other pairs of sides. Thus we would have finally the awe-inspiring (or perhaps, ahem, awe-ful!) expression
$$\begin{aligned}
u = &-\sum_{\substack{(\ell,m,n)\neq(0,0,0)\\ \ell,m,n\neq 2}} \frac{64(1-(-1)^\ell)(1-(-1)^m)(1-(-1)^n)}{\pi^5(\ell^2+m^2+n^2)(\ell^2-4)(m^2-4)(n^2-4)}\cos\ell\pi x\cos m\pi y\cos n\pi z \\
&+8\sum_{\ell=1}^\infty\frac{1+(-1)^\ell}{\pi^2\ell^2}\cos\ell\pi x + 8\sum_{m=1}^\infty\frac{1+(-1)^m}{\pi^2 m^2}\cos m\pi y + 8\sum_{n=1}^\infty\frac{1+(-1)^n}{\pi^2 n^2}\cos n\pi z.
\end{aligned}$$

[The above solution would not be unique since adding any constant to it will give another solution to the original problem. This constant could be fixed by giving another condition such as the condition $u(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = 0$ which we had in problem 1 of homework 10.]

Unfortunately the above procedure fails to actually give a solution, partly because the problem as stated has in fact no solution (your instructor apparently failed to notice this somehow). The reason for this is easy to see once one thinks about it for a bit. We have the problem
$$\nabla^2 u = \sin 2\pi x \sin 2\pi y \sin 2\pi z, \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial Q} = 1.$$

Let us integrate $\nabla^2 u$ over Q and apply the divergence theorem:

$$\int_Q \nabla^2 u \, d\mathbf{x} = \int_Q \nabla \cdot \nabla u \, d\mathbf{x}$$
$$= \int_{\partial Q} \mathbf{n} \cdot \nabla u \, d\mathbf{x} = \int_{\partial Q} \frac{\partial u}{\partial n} \, dS$$
$$= \int_{\partial Q} 1 \, dS = 6;$$

but we have also

$$\int_Q \nabla^2 u \, d\mathbf{x} = \int_Q \sin 2\pi x \sin 2\pi y \sin 2\pi z \, d\mathbf{x}$$
$$= 0,$$

a contradiction. In other words, the problem

$$\nabla^2 u = f, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial Q} = g$$

must satisfy the consistency condition

$$\int_Q f \, d\mathbf{x} = \int_{\partial Q} g \, dS$$

in order to have a solution. (This is an extension of the condition $(f, e_I) = 0$ for I such that $\lambda_I = 0$ which we derive in the Appendix.) Since the given f and g do not satisfy this condition, this problem has no solution as stated. We apologise.

2. Using Fourier transforms in space, solve the problem on $(0, +\infty) \times \mathbf{R}^3$

$$\frac{\partial u}{\partial t} = \nabla^2 u + \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}}, \quad u|_{t=0} = 0.$$

[It is worth considering what would happen if a factor other than 4 were used in the exponent above; but the calculations would become far more involved.]

This problem is actually quite easy (particularly compared to the previous one!). First we recall (see the solutions to Homework 10) that on $\mathbf{R}^3$

$$\mathcal{F}\left[e^{-a|\mathbf{x}|^2}\right](\mathbf{k}) = \left(\frac{\pi}{a}\right)^{\frac{3}{2}} e^{-\frac{\pi^2|\mathbf{k}|^2}{a}};$$

an exactly analogous result holds for the inverse Fourier transform:

$$\mathcal{F}^{-1}\left[e^{-a|\mathbf{k}|^2}\right](\mathbf{x}) = \left(\frac{\pi}{a}\right)^{\frac{3}{2}} e^{-\frac{\pi^2|\mathbf{x}|^2}{a}}$$

(this can either be seen by turning the first result above backwards – i.e., by replacing a by $\frac{\pi^2}{a}$ and moving the multiplicative factor to the left-hand side – or by noting that for a real-valued function f

$$\mathcal{F}^{-1}[f](\mathbf{x}) = \overline{\mathcal{F}[f](\mathbf{x})}.)$$

Thus, assuming that u and sufficiently many of its derivatives have Fourier transforms, we may take the Fourier transform of the above problem to obtain

$$\frac{\partial \hat{u}}{\partial t} = -4\pi^2|\mathbf{k}|^2\hat{u} + \frac{1}{\sqrt{t}}(4\pi t)^{\frac{3}{2}} e^{-4\pi^2|\mathbf{k}|^2 t}, \quad \hat{u}|_{t=0} = 0;$$

multiplying the first equation by $e^{4\pi^2|\mathbf{k}|^2t}$ and rearranging gives

$$\frac{\partial}{\partial t}\left(e^{4\pi^2|\mathbf{k}|^2t}\hat{u}\right) = (4\pi)^{\frac{3}{2}}t$$
$$\hat{u} = 4\pi^{\frac{3}{2}}t^2e^{-4\pi^2|\mathbf{k}|^2t},$$

so, taking the inverse Fourier transform, we obtain

$$u = 4\pi^{\frac{3}{2}}t^2\left(\frac{\pi}{4\pi^2t}\right)^{\frac{3}{2}}e^{-\frac{x^2}{4t}} = 4\pi^{\frac{3}{2}}t^2t^{-\frac{3}{2}}(4\pi)^{-\frac{3}{2}}e^{-\frac{x^2}{4t}} = \frac{\sqrt{t}}{2}e^{-\frac{x^2}{4t}}.$$

If the coefficient in the exponent in the original problem were not $\frac{1}{4}$, the Gaussian factors would not cancel, but we would still be able to integrate because of the t factor. (Note that if we were working in any dimension other than 3 the t factor would become t^α for some $\alpha \neq 1$, and we would not in general be able to integrate in closed form.)

3. [Optional.] By analogy with our derivation in class of the eigenfunctions of the Laplacian on the cylinder C, derive the eigenfunctions and eigenvalues of the Laplacian on the disk $D = \{(r,\theta)|r < 1\}$ satisfying Dirichlet boundary conditions. Now consider the wave equation on D with Dirichlet boundary conditions:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{\partial D} = 0.$$

Find the set of all possible frequencies f such that the above problem has a solution of the form $e^{2\pi i f t}\Phi(r,\theta)$ for some function $\Phi(r,\theta)$. These are the *natural frequencies* for a circular drumhead: they are the frequencies at which it can oscillate continuously (ignoring losses due to heating in the drumhead and the transmitting of energy from the drumhead to the air to create the sound waves which we actually hear, of course). Any forced motion at another frequency would rapidly die out.

[Sketch.] Since the Laplacian in polar coordinates is the same as the Laplacian in cylindrical coordinates except that it lacks the $\frac{\partial^2}{\partial z^2}$ term, we see the the eigenfunctions for the Laplacian in polar coordinates are simply

$$J_m(\lambda_{mi}\rho)\cos m\phi, \quad J_m(\lambda_{mi}\rho)\sin m\phi,$$

with eigenvalues

$$\lambda = -\lambda_{mi}^2.$$

(It is worthwhile to derive these results by working directly from separation of variables in polar coordinates.) Now suppose that we have a solution u to the above problem which is of the form $u = e^{2\pi i f t}\Phi(r,\theta)$; substituting in, we obtain for Φ the problem

$$-4\pi^2 f^2\Phi = \nabla^2\Phi, \quad \Phi|_{\partial D} = 0.$$

The left-hand equation here is known as the *Helmholtz equation*, and is easily seen to be simply the eigenvalue problem for the Laplacian on the unit disk. By the foregoing, then, we see that we must have, for some m, i,

$$-4\pi^2 f^2 = -\lambda_{mi}^2,$$
$$f = \pm\frac{1}{2\pi}\lambda_{mi}.$$

[We now have enough background to appreciate at least part of the following question, which arises in the study of inverse problems, and was posed by the mathematician Mark Kac: Can one hear the shape of a drum? More precisely, suppose that for some region D in the plane we are given the set of all possible frequencies f for which the wave equation on D possesses solutions with the single frequency f, i.e., possesses solutions of the form above, $e^{2\pi i f t}\Phi(r,\theta)$. The question then is whether this set of frequencies uniquely determines D. (More generally, one considers a so-called Riemannian manifold and the generalised Laplacian on it.) The answer, as the author saw it put in a course prospectus when he was at Cambridge a long time ago, is No, but Almost Yes. Unfortunately that about exhausts the knowledge of the current author on the subject!]

Appendix: Green's functions in the presence of zero eigenvalues. Let us suppose that the eigenvalue problem

$$\nabla^2 u = \lambda u, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial D} = 0$$

on some region D has a zero eigenvalue, i.e., that there is a nonzero function e which satisfies the above with $\lambda = 0$. We would like to find a Green's function in this case; but the standard formula would involve a division by zero, as noted in the solutions to problem 1 above. To derive an appropriate formula for the solution in this case, we go back to first principles. Suppose that $\{\mathbf{e}_I\}$ is a complete set of eigenfunctions for the above problem with $(\mathbf{e}_I, \mathbf{e}_I) = 1$, with corresponding eigenvalues λ_I. Let $\mathbf{I}_0 = \{I | \lambda_I = 0\}$. Let us consider first the homogeneous problem

$$\nabla^2 u = f, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial D} = 0.$$

If we write $f = \sum_I b_I \mathbf{e}_I$, $u = \sum_I a_I \mathbf{e}_I$, then substituting in gives

$$\sum_I \lambda_I a_I \mathbf{e}_I = \sum_I b_I \mathbf{e}_I.$$

Since $\{e_I\}$ is an orthogonal set of nonzero functions, this gives for $I \notin \mathbf{I}_0$ that $a_I = \frac{1}{\lambda_I} b_I$, which is the same as we had before. If, however, $I \in \mathbf{I}_0$, then this relation gives instead $b_I = 0$; i.e., it becomes a restriction on the functions f for which the problem has a solution, rather than information about the solution. We assume that f is such that $b_I = (f, e_I) = 0$ for all $I \in \mathbf{I}_0$, so that this condition is satisfied. We note also that a_I is undetermined for $I \in \mathbf{I}_0$. Thus we may write

$$u = \sum_{I \notin \mathbf{I}_0} \frac{1}{\lambda_I} b_I \mathbf{e}_I + \sum_{I \in \mathbf{I}_0} a_I \mathbf{e}_I.$$

The second sum can only be determined by auxiliary information (for example, in problem 1 of homework 10 the condition $u(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 0$ allowed us to determine that part of the sum). Thus here we drop it and consider only the first term. We have, as in our previous derivation of the Green's function, that (assuming as usual that we may interchange integration and summation)

$$u = \sum_{I \notin \mathbf{I}_0} \frac{1}{\lambda_I} \int_D f(\mathbf{x}') \overline{\mathbf{e}_I(\mathbf{x}')}\, d\mathbf{x}' \mathbf{e}_I(\mathbf{x}) = \int_D f(\mathbf{x}') \left[\sum_{I \notin \mathbf{I}_0} \frac{\mathbf{e}_I(\mathbf{x}) \overline{\mathbf{e}_I(\mathbf{x}')}}{\lambda_I} \right] d\mathbf{x}'.$$

This suggests that we should take as the Green's function

$$G(\mathbf{x}, \mathbf{x}') = -\sum_{I \notin \mathbf{I}_0} \frac{\mathbf{e}_I(\mathbf{x}) \overline{\mathbf{e}_I(\mathbf{x}')}}{\lambda_I}.$$

This satisfies a homogeneous Neumann condition by construction. Now if we proceed formally as in our original derivation of a Green's function, we may write

$$\nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}') = -\sum_{I \notin \mathbf{I}_0} \mathbf{e}_I(\mathbf{x}) \overline{\mathbf{e}_I(\mathbf{x}')};$$

since for any f satisfying $(f, \mathbf{e}_I) = 0$ when $I \in \mathbf{I}_0$ we have

$$f(x) = \sum_I (f, \mathbf{e}_I) e_I = \sum_{I \notin \mathbf{I}_0} \int_D f(\mathbf{x}') \overline{\mathbf{e}_I(\mathbf{x}')}\, d\mathbf{x}' \mathbf{e}_I(x)$$

$$= \int_D f(\mathbf{x}') \left[\sum_{I \notin \mathbf{I}_0} \mathbf{e}_I(\mathbf{x}) \mathbf{e}_I(\mathbf{x}') \right] d\mathbf{x}',$$

we see that as long as we restrict to functions satisfying the above condition we may write $\nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}')$, as before. Examining the proof of the relation used in the solution of problem 1 above in this case, we see that it also holds if we assume that u likewise satisfies $(u, \mathbf{e}_I) = 0$ for $I \in \mathbf{I}_0$. This justifies the solution given above.