APM 346, Homework 11. Due Monday, August 5, at 6.00 AM EDT. To be marked completed/not completed.

1. Using the eigenfunctions derived in homework 10, problem 1, construct the Green's function on Q satisfying

$$\nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}'), \quad \left.\frac{\partial G}{\partial n}\right|_{\mathbf{x} \in \partial Q} = 0,$$

and use it to find a series expansion for the solution to the following problem on Q:

$$\nabla^2 u = \sin 2\pi x \sin 2\pi y \sin 2\pi z, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial Q} = 1.$$

[The following question is worth considering: What would happen if we replaced $\sin 2\pi x \sin 2\pi y \sin 2\pi z$ by $\sin \pi x \sin \pi y \sin \pi z$ above?]

2. Using Fourier transforms in space, solve the problem on $(0, +\infty) \times \mathbf{R}^3$

$$\frac{\partial u}{\partial t} = \nabla^2 u + \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}}, \quad u|_{t=0} = 0.$$

[It is worth considering what would happen if a factor other than 4 were used in the exponent above; but the calculations would become far more involved.]

3. [Optional.] By analogy with our derivation in class of the eigenfunctions of the Laplacian on the cylinder C, derive the eigenfunctions and eigenvalues of the Laplacian on the disk $D = \{(r, \theta) | r < 1\}$ satisfying Dirichlet boundary conditions. Now consider the wave equation on D with Dirichlet boundary conditions:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{\partial D} = 0.$$

Find the set of all possible frequencies f such that the above problem has a solution of the form $e^{2\pi i f t} \Phi(r, \theta)$ for some function $\Phi(r, \theta)$. These are the *natural frequencies* for a circular drumhead: they are the frequencies at which it can oscillate continuously (ignoring losses due to heating in the drumhead and the transmitting of energy from the drumhead to the air to create the sound waves which we actually hear, of course). Any forced motion at another frequency would rapidly die out.

[We now have enough background to appreciate at least part of the following question, which arises in the study of inverse problems, and was posed by the mathematician Mark Kac: Can one hear the shape of a drum? More precisely, suppose that for some region D in the plane we are given the set of all possible frequencies f for which the wave equation on D possesses solutions with the single frequency f, i.e., possesses solutions of the form above, $e^{2\pi i f t} \Phi(r, \theta)$. The question then is whether this set of frequencies uniquely determines D. (More generally, one considers a so-called Riemannian manifold and the generalised Laplacian on it.) The answer, as the author saw it put in a course prospectus when he was at Cambridge a long time ago, is No, but Almost Yes. Unfortunately that about exhausts the knowledge of the current author on the subject!]