APM 346, Homework 10. Due Monday, July 29, at 6.00 AM EDT. To be marked completed/not completed.

1. Starting from separation of variables, give the series expansion to the solution for the following problem in terms of an appropriate set of eigenfunctions of the Laplacian on the unit cube $Q = \{(x, y, z)|0 \le x, y, z \le 1\}$:

$$\nabla^2 u = \begin{cases} 1, & 0 \le z < \frac{1}{2} \\ -1, & \frac{1}{2} < z \le 1 \end{cases}, \quad \partial_\nu u|_{\partial Q} = 0, \quad u(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 0,$$

where ∂_ν denotes the outward normal derivative on the surface (e.g., on the surface $\partial Q \cap \{z = 0\}$, it is $-\frac{\partial}{\partial z}$).

We begin by finding the eigenfunctions of the Laplacian on Q appropriate to the given boundary conditions. (The last condition $u(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 0$ is a condition on the solution, not the eigenfunctions, and will be dealt with at the end.) We shall look as usual for separated eigenfunctions; thus we seek functions $u = X(x)Y(y)Z(z)$ and numbers λ satisfying

$$\nabla^2 u = \lambda u, \quad \partial_\nu u|_{\partial Q} = 0;$$

now substituting $u = X(x)Y(y)Z(z)$ into the first equation and dividing through by u (since we assume that u, as an eigenfunction, is not identically zero), we have as usual the equation

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \lambda. \tag{1}$$

Now we need to determine how the boundary conditions are to be implemented in terms of X, Y, and Z. Now the boundary ∂Q of Q has six parts, which lie in the planes $z = 0$, $z = 1$, $x = 0$, $x = 1$, $y = 0$, $y = 1$; since ∂_ν is the unit outward normal derivative on the boundary of Q, we see that on the plane $z = 0$, $\partial_\nu = -\frac{\partial}{\partial z}$, while on $z = 1$ we have $\partial_\nu = \frac{\partial}{\partial z}$; thus the parts of the boundary condition $\partial_\nu u|_{\partial Q} = 0$ corresponding to the top and bottom surfaces of the cube are

$$X(x)Y(y)\left(-Z'(0)\right) = 0, \quad X(x)Y(y)Z'(1) = 0,$$

i.e., $X(x)Y(y)Z'(0) = X(x)Y(y)Z'(1) = 0$ for all x and y. Since X and Y are not identically zero, we conclude that $Z'(0) = Z'(1) = 0$. Analogously, the boundary conditions on the other sides of the cube give $X'(0) = X'(1) = 0$, $Y'(0) = Y'(1) = 0$, and we thus have in addition to (1) the boundary conditions

$$X'(0) = X'(1) = Y'(0) = Y'(1) = Z'(0) = Z'(1) = 0.$$

From these we see as usual (since the derivative of a linear combination of exponentials is still a linear combination of exponentials) that X, Y, and Z must all be oscillatory; thus $\frac{X''}{X}$, $\frac{Y''}{Y}$, $\frac{Z''}{Z} < 0$, so we may write

$$X'' = -\lambda_1^2 X, \quad Y'' = -\lambda_2^2 Y, \quad Z'' = -\lambda_3^2 Z$$

(note that we do not yet know what the λ_i are since the boundary conditions are not the homogeneous Dirichlet conditions we have met previously; in other words, we cannot just directly write $\lambda_1 = \ell\pi$, etc.). Let us consider the problem for X:

$$X'' = -\lambda_1^2 X, \quad X'(0) = X'(1) = 0.$$

From the equation, we have

$$X = a \cos \lambda_1 x + b \sin \lambda_1 x,$$

whence the boundary conditions give

$$X'(0) = -\lambda_1 b = 0, \quad X'(1) = -a\lambda_1 \sin \lambda_1 + b\lambda_1 \cos \lambda_1 = 0;$$

the first gives either $\lambda_1 = 0$, in the which case $X = a$ is constant, or $b = 0$; in the first case the second boundary condition is satisfied automatically, while in the second case ($\lambda_1 \neq 0$, $b = 0$) it gives

$$a\lambda_1 \sin\lambda_1 = 0,$$

so (since $a \neq 0$ as $X \neq 0$, and $\lambda_1 \neq 0$ by assumption) we must have $\lambda_1 = \ell\pi$, $\ell \in \mathbf{Z}$, $\ell > 0$, as before. Thus we have two separate cases: either $X = a$ or $X = a\cos\ell\pi x$, $\ell \in \mathbf{Z}$, $\ell > 0$. Clearly, we may combine these two cases; dropping the arbitrary constant a, we may write

$$X = \cos\ell\pi x, \quad \ell \in \mathbf{Z}, \ell \geq 0.$$

Similar logic clearly applies also to Y and Z, so we have

$$\begin{aligned} Y &= \cos m\pi y, \quad m \in \mathbf{Z}, m \geq 0, \\ Z &= \cos n\pi z, \quad n \in \mathbf{Z}, n \geq 0, \end{aligned}$$

and we have finally the eigenfunctions

$$\mathbf{e}_{\ell mn} = \cos\ell\pi x \cos m\pi y \cos n\pi z, \quad \ell, m, n \in \mathbf{Z}, \ell, m, n \geq 0,$$

while the corresponding eigenvalues are

$$\lambda_{\ell mn} = -\pi^2\left(\ell^2 + m^2 + n^2\right).$$

Note that $\lambda_{000} = 0$, i.e., we have a zero eigenvalue; this is because the constant function satisfies the boundary condition in this case. This will create some extra wrinkles in our solution, one of which is obvious while one is less so, as we shall see shortly.

We note that the set $\{\cos\ell\pi x\}_{\ell=0}^{\infty}$ is complete on $[0,1]$; this can be shown in a way similar to that by which we showed $\{\sin\ell\pi x\}_{\ell=1}^{\infty}$ complete on $[0,1]$: if $f : [0,1] \to \mathbf{R}^1$ is any suitable function, then we may extend it to $[-1,1]$ by requiring it to be *even*, i.e., we may define a new function

$$f^* : [-1,1], \quad f^*(x) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x \leq 0 \end{cases};$$

since $\{\cos\ell\pi x, \sin\ell\pi x\}_{\ell=0}^{\infty}$ is complete on $[-1,1]$, we may expand f^* in a aeries in $\cos\ell\pi x$ and $\sin\ell\pi x$; but since f^* is even, all of the coefficients for the $\sin\ell\pi x$ terms vanish, meaning that f^* can be written in a series

$$f^* = \sum_{\ell=0}^{\infty} a_\ell \cos\ell\pi x$$

on $[-1,1]$. But from this it follows that on $[0,1]$ we have the series

$$f = \sum_{\ell=0}^{\infty} a_\ell \cos\ell\pi x,$$

meaning that $\{\cos\ell\pi x\}_{\ell=0}^{\infty}$ is complete on $[0,1]$, as desired. By standard logic, it follows that the set of eigenfunctions $\{\mathbf{e}_{\ell mn}\}_{\ell,m,n=0}^{\infty}$ is complete on Q.

We may now proceed as usual to solve the equation. We begin by expanding the right-hand side of the given Poisson equation in terms of the above basis of eigenfunctions. Thus let

$$g(x,y,z) = \begin{cases} 1, & 0 \leq z < \frac{1}{2} \\ -1, & \frac{1}{2} < z \leq 1 \end{cases};$$

then we may write

$$g(x,y,z) = \sum_{\ell,m,n=0}^{\infty} a_{\ell mn} \cos \ell\pi x \cos m\pi y \cos n\pi z.$$

To write out a formula for the $a_{\ell mn}$, we need to determine the normalisation constants for the $\mathbf{e}_{\ell mn}$. Now

$$\int_0^1 \cos^2 \ell\pi x \, dx = \begin{cases} 1, & \ell = 0 \\ \frac{1}{2}, & \ell \neq 0 \end{cases};$$

if we denote this quantity by N_ℓ, then we may write

$$\int_Q \mathbf{e}_{\ell mn}^2(x,y,z)\, dV = N_\ell N_m N_n.$$

Thus we may write the coefficients $a_{\ell mn}$ in the above expansion as

$$\begin{aligned} a_{\ell mn} &= \frac{1}{N_\ell N_m N_n} \int_Q g(x,y,z)\mathbf{e}_{\ell mn}\, dV = \frac{1}{N_\ell N_m N_n} \int_0^1 \int_0^1 \int_0^1 g(x,y,z) \cos \ell\pi x \cos m\pi y \cos n\pi z \, dz\, dy\, dx \\ &= \frac{1}{N_\ell N_m N_n} \int_0^1 \cos \ell\pi x \, dx \int_0^1 \cos m\pi y \, dy \int_0^1 g \cos n\pi z \, dz, \end{aligned}$$

where we have used the fact that g depends only on z. Now we may write

$$\int_0^1 \cos \ell\pi x \, dx = (1, \cos \ell\pi x) = \begin{cases} 1, & \ell = 0 \\ 0, & \ell \neq 0 \end{cases},$$

since both $1 = \cos 0\pi x$ is an element of the orthogonal set $\{\cos \ell\pi x\}_{\ell=0}^{\infty}$. From this, we see that $a_{\ell mn} = 0$ unless $\ell = m = 0$. Further, we see that

$$\int_0^1 g \cos 0\pi z \, dz = \int_0^1 g \, dz = 0,$$

so that $a_{000} = 0$, while if $n \neq 0$

$$\begin{aligned} a_{00n} &= 2\int_0^1 g \cos n\pi z \, dz = 2\left(\int_0^{\frac{1}{2}} \cos n\pi z \, dz - \int_{\frac{1}{2}}^1 \cos n\pi z \, dz \right) \\ &= 2\left(\frac{1}{n\pi} \sin n\pi z \bigg|_0^{\frac{1}{2}} - \frac{1}{n\pi} \sin n\pi z \bigg|_{\frac{1}{2}}^1 \right) = \frac{4}{n\pi} \sin \frac{n\pi}{2}. \end{aligned}$$

Thus we have finally

$$g(x,y,z) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi}{2} \cos n\pi z.$$

Now we assume that the solution u to $\nabla^2 u = g$ may be expanded in the basis $\{\mathbf{e}_{\ell mn}\}_{\ell,m,n=0}^{\infty}$ as

$$u = \sum_{\ell,m,n=0}^{\infty} b_{\ell mn} \cos \ell\pi x \cos m\pi y \cos n\pi z;$$

substituting this in, and using the series expansion for g above, we have

$$\sum_{\ell,m,n=0}^{\infty} \lambda_{\ell mn} b_{\ell mn} \mathbf{e}_{\ell mn} = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi}{2} \cos n\pi z;$$

from this we see, first of all, that

$$\lambda_{000} b_{000} = a_{000} = 0;$$

but since $\lambda_{000} = 0$, this tells us nothing about b_{000}. Thus b_{000} is not determined by the boundary conditions on u. We note also that had g been such that $a_{000} \neq 0$ – which, unravelling everything, amounts to saying $\int_Q g\, dV \neq 0$ – then the above equation would become

$$\lambda_{000} b_{000} = 0 = a_{000} \neq 0,$$

which has no solution. If we recall our abstract formula for the solution to Poisson's equation $nabla^2 u = g$,

$$u = \sum_I \frac{1}{\lambda_I} \frac{(g, e_I)}{(e_I, e_I)} e_I,$$

we see that this is exactly the condition that $(g, e_I) = 0$ for all I for which λ_I vanishes, while also the coefficients in the series for u corresponding to such I are undetermined. These are common difficulties when the Laplacian has a zero eigenvalue.

Proceeding to the nonzero eigenvalues, we see that $b_{\ell mn} = 0$ unless $\ell = m = 0$, while for $n \neq 0$

$$b_{00n} = \frac{1}{\lambda_{\ell mn}} \frac{4}{n\pi} \sin\frac{n\pi}{2} = -\frac{4}{n^3\pi^3} \sin\frac{n\pi}{2}.$$

Thus we have the series solution

$$u = b_{000} - \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3} \sin\frac{n\pi}{2} \cos n\pi z.$$

To determine b_{000}, we apply the final condition, noting that $\sin\frac{n\pi}{2}\cos\frac{n\pi}{2} = \frac{1}{2}\sin n\pi = 0$ for all $n \in \mathbf{Z}$:

$$u(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = b_{000} - \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3} \sin\frac{n\pi}{2} \cos\frac{n\pi}{2} = b_{000} = 0,$$

so that finally we have the solution

$$u(x, y, z) = -\sum_{n=1}^{\infty} \frac{4}{n^3\pi^3} \sin\frac{n\pi}{2} \cos n\pi z.$$

2. Compute the Fourier transforms of the following functions:

$$f(x) = \begin{cases} 1, & x \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}.$$

$$f(x) = \begin{cases} 1 - |x|, & x \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}.$$

$$f(r, \theta, \phi) = \begin{cases} 1, & r \leq 1 \\ 0, & \text{otherwise} \end{cases}.$$

$$f(x) = e^{-ax^2}, \quad a \in \mathbf{R}, a > 0.$$

$$f(r, \theta, \phi) = e^{-ar^2}, \quad a \in \mathbf{R}, a > 0.$$

$$f(x) = xe^{-ax^2}, \quad a \in \mathbf{R}, a > 0.$$

[For the fifth of these, it may be simpler to change to rectangular coordinates.]

We take these one by one:

$$\begin{aligned}\mathcal{F}[f](k) &= \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\,dx = \int_{-1}^{1} e^{-2\pi ikx}\,dx = \frac{i}{2\pi k}e^{-2\pi ikx}\Big|_{-1}^{1} \\ &= \frac{i}{2\pi k}\left(e^{-2\pi ik} - e^{2\pi ik}\right) = \frac{\sin 2\pi k}{\pi k}.\end{aligned}$$

The above calculation only works for $k \neq 0$; but for $k = 0$ we have clearly $\mathcal{F}[f](0) = 2$, which is the limit of the above function as $k \to 0$. Thus we have

$$\mathcal{F}[f](k) = \begin{cases} 2, & k = 0 \\ \frac{\sin 2\pi k}{\pi k}, & k \neq 0 \end{cases}.$$

This function of k is closely related to the so-called sinc function, which is useful in many different places. We shall typically just write it as $\frac{\sin 2\pi k}{\pi k}$, with the understanding that its value at $k = 0$ is taken to be 2. (We note that with this definition it is actually an analytic function of k with a power series expansion convergent on the entire real line, or complex plane.)

Next, we have

$$\mathcal{F}[f](k) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\,dx = \int_{-1}^{1} (1 - |x|)\,e^{-2\pi ikx}\,dx.$$

To compute this integral, we note that for $k \neq 0$

$$\int xe^{-2\pi ikx}\,dx = -\frac{1}{2\pi ik}xe^{-2\pi ikx} + \frac{1}{2\pi ik}\int e^{-2\pi ikx}\,dx = \left(-\frac{1}{2\pi ik}x + \frac{1}{4\pi^2k^2}\right)e^{-2\pi ikx} + C,$$

while when $k = 0$

$$\int xe^{-2\pi ikx}\,dx = \int x\,dx = \frac{1}{2}x^2 + C.$$

Thus the above integrals become

$$\int_{-1}^{1} (1 - |x|)\,e^{-2\pi ikx}\,dx = \int_{-1}^{1} e^{-2\pi ikx}\,dx + \int_{-1}^{0} xe^{-2\pi ikx}\,dx - \int_{0}^{1} xe^{-2\pi ikx}\,dx;$$

the first of these is just $\frac{\sin 2\pi k}{\pi k}$, while the second two give

$$\begin{aligned}&\left(-\frac{1}{2\pi ik}x + \frac{1}{4\pi^2k^2}\right)e^{-2\pi ikx}\Big|_{-1}^{0} - \left(-\frac{1}{2\pi ik}x + \frac{1}{4\pi^2k^2}\right)e^{-2\pi ikx}\Big|_{0}^{1} \\ &\qquad= \frac{1}{4\pi^2k^2} - \left(\frac{1}{2\pi ik} + \frac{1}{4\pi^2k^2}\right)e^{2\pi ik} - \left(\left(-\frac{1}{2\pi ik} + \frac{1}{4\pi^2k^2}\right)e^{-2\pi ik} - \frac{1}{4\pi^2k^2}\right) \\ &\qquad= \frac{1}{2\pi^2k^2}\left(1 - \frac{1}{2}\left(e^{2\pi ik} + e^{-2\pi ik}\right)\right) - \frac{1}{\pi k}\frac{1}{2i}\left(e^{2\pi ik} - e^{-2\pi ik}\right) \\ &\qquad= \frac{1}{2\pi^2k^2}(1 - \cos 2\pi k) - \frac{\sin 2\pi k}{\pi k},\end{aligned}$$

in the case that $k \neq 0$; when $k = 0$, they give simply

$$\int_{-1}^{0} x\,dx - \int_{0}^{1} x\,dx = \frac{1}{2}x^2\Big|_{-1}^{0} - \frac{1}{2}x^2\Big|_{0}^{1} = -1,$$

which is seen to be the limiting value of the above expression as $k \to 0$. Taking it to have this value at $k = 0$ (as we did with $\frac{\sin 2\pi k}{\pi k}$ above), we have finally

$$\begin{aligned}\mathcal{F}[f](k) &= \frac{\sin 2\pi k}{\pi k} + \frac{1}{2\pi^2k^2}(1 - \cos 2\pi k) - \frac{\sin 2\pi k}{\pi k} \\ &= \frac{1}{2\pi^2k^2}(1 - \cos 2\pi k).\end{aligned}$$

(We note that, defining this function to have its limiting value at $k = 0$, it is also analytic.)

Proceeding, we have

$$\mathcal{F}[f](\mathbf{k}) = \int_{\mathbf{R}^3} f(r, \theta, \phi) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}.$$

Fix some $\mathbf{k} \in \mathbf{R}^3$. Now since f is spherically symmetric, we may assume that our spherical coordinate system (r, θ, ϕ) is such that in it $\mathbf{k} = (k, 0, 0)$, i.e., that $\mathbf{k}$ points along the positive z axis. In this case, $\mathbf{k} \cdot \mathbf{x} = kr\cos\theta$, and the above integral may be written

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 e^{-2\pi i kr\cos\theta} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

which may be evaluated as

$$\begin{aligned}
2\pi \int_0^1 \frac{1}{2\pi i k} r e^{-2\pi i k r\cos\theta}\bigg|_0^{\pi} dr &= 2\pi \int_0^1 \frac{1}{2\pi i k} r\left(e^{2\pi i k r} - e^{-2\pi i k r}\right) dr = 2\pi \int_0^1 \frac{r\sin 2\pi k r}{\pi k}\, dr \\
&= \frac{2}{k}\left(-r\frac{\cos 2\pi k r}{2\pi k}\bigg|_0^1 + \frac{1}{2\pi k}\int_0^1 \cos 2\pi k r\, dr\right) \\
&= \frac{2}{k}\left(-\frac{\cos 2\pi k}{2\pi k} + \frac{1}{4\pi^2 k^2}\sin 2\pi k r\bigg|_0^1\right) = \frac{2}{k}\left(-\frac{\cos 2\pi k}{2\pi k} + \frac{\sin 2\pi k}{4\pi^2 k^2}\right) \\
&= \frac{1}{2\pi^2 k^3}\left(-2\pi k \cos 2\pi k + \sin 2\pi k\right),
\end{aligned}$$

for $k \neq 0$, while if $k = 0$ it is clearly just $\frac{4}{3}\pi$, the volume of the unit sphere; and we note that this is just the limit of the above expression as $k \to 0$:

$$\begin{aligned}
\frac{1}{2\pi^2 k^3}\left(-2\pi k\cos 2\pi k + \sin 2\pi k\right) &= \frac{1}{2\pi^2 k^3}\left(-2\pi k + \pi k (2\pi k)^2 - \cdots + 2\pi k - \frac{1}{6}(2\pi k)^3 + \cdots\right) \\
&= \frac{1}{2\pi^2 k^3}\left(4\pi^3 k^3 - \frac{4}{3}\pi^3 k^3 + \cdots\right) = \frac{\frac{8}{3}\pi^3 k^3 + \cdots}{2\pi^2 k^3} = \frac{4}{3}\pi + \cdots,
\end{aligned}$$

where $\cdots$ indicates terms of order in k higher than those preceding. This expression thus clearly approaches $\frac{4}{3}\pi$ as $k \to 0$, as claimed.

Continuing with fortitude, we have, noting the Gaussian integral

$$\int_{\mathbf{R}^1} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}$$

(which holds for all complex a with $\Re a > 0$)

$$\begin{aligned}
\mathcal{F}[f](k) &= \int_{-\infty}^{\infty} e^{-ax^2} e^{-2\pi i k x}\, dx \\
&= \int_{-\infty}^{\infty} e^{-a\left(x + \frac{\pi i k}{a}\right)^2 - \frac{\pi^2 k^2}{a}}\, dx = e^{-\frac{\pi^2 k^2}{a}} \int_{-\infty}^{\infty} e^{-a\left(x+\frac{\pi i k}{a}\right)^2}\, dx \\
&= \sqrt{\frac{\pi}{a}} e^{-\frac{\pi^2 k^2}{a}},
\end{aligned}$$

where we have used the substitution $u = x + \frac{\pi i k}{a}$ in the last equality. (This can be justified more rigorously in the context of complex variable theory by thinking of adjusting the contour $z = t$ to the contour $z = t + \frac{\pi i k}{a}$ bit by bit, and noting that the integrand rapidly goes to zero as $t \to \pm\infty$ along either contour.) We note that the width of the Gaussian giving the Fourier transform is proportional to the reciprocal of the width of the original Gaussian; this is a manifestation of the celebrated *uncertainty principle*, which is probably best known from quantum mechanics but can also be formulated as a theorem on Fourier transforms (since, we note for those who have seen some quantum mechanics, the momentum-space representation of the wavefunction is essentially just the Fourier transform of its position-space representation).

Continuing, and using the hint, we have

$$\begin{aligned}\mathcal{F}[f](\mathbf{k}) &= \int_{\mathbf{R}^3} e^{-ar^2} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} \\ &= \int_{\mathbf{R}^3} e^{-a\left(x^2+y^2+z^2\right)} e^{-2\pi i(k_1 x + k_2 y + k_3 z)}\, d\mathbf{x},\end{aligned}$$

which is easily seen to be a product of three transforms of Gaussians; in other words, we have

$$\mathcal{F}[f](\mathbf{k}) = \left(\frac{\pi}{a}\right)^{\frac{3}{2}} e^{-\frac{\pi^2\left(k_1^2+k_2^2+k_3^2\right)}{a}} = \left(\frac{\pi}{a}\right)^{\frac{3}{2}} e^{-\frac{\pi^2|\mathbf{k}|^2}{a}}.$$

For the final Fourier transform, we could proceed directly, but that would be quite a nuisance; instead we use a property of the Fourier transform to write

$$\begin{aligned}\mathcal{F}[xe^{-ax^2}](k) = \mathcal{F}\left[-\frac{1}{2a}\frac{d}{dx}\left(e^{-ax^2}\right)\right](k) &= -\frac{1}{2a}2\pi i k\sqrt{\frac{\pi}{a}}e^{-\frac{\pi^2 k^2}{a}} \\ &= -ik\left(\frac{\pi}{a}\right)^{\frac{3}{2}} e^{-\frac{\pi^2 k^2}{a}}.\end{aligned}$$

This formula is related to the properties of the so-called *Hermite polynomials* discussed in section 5.2.8 of the textbook.