Effective universal coverings and local minima of the length functional on loop spaces.

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Abstract. In this paper we describe a construction of effective universal coverings of Riemannian manifolds (and, more generally, of path metric spaces). We give two applications of this construction.

To describe the first application let M^n be a simply connected closed Riemannian manifold, and $\Omega_x M^n$ denote the space of all loops based at a point $x \in M^n$. We demonstrate that if the length functional on $\Omega_x M^n$ has a non-trivial very deep local minimum, then it has many deep local minima.

The second application is a quantitative version of a theorem proven by M. Anderson. To state this theorem assume that M^n is a closed n-dimensional manifold with volume $\geq v > 0$, diameter d and Ricci curvature $\geq -(n-1)$. Then the theorem asserts that there exist explicit positive $\epsilon = \epsilon(n, v, d)$ and $N = N(n, v, d)$ such that for each closed curve γ of length smaller than ϵ one of its first N iterates is contractible. In the present paper we find an explicit upper bound for the length of curves in a homotopy contracting one of the first N iterates of γ.

0. Introduction.

0.1. Recall that the standard construction of the universal covering of a topological space goes as follows: One takes all paths $\gamma \in \Omega_x X$ starting at a fixed base point $x \in X$ and identifies all pairs of paths $\gamma_1, \gamma_2 \in \Omega_x X$ that have a common endpoint and together form a contractible loop $\gamma_1 * \gamma_2^{-1}$ based at x (cf. [M]). Assume that X is a Riemannian manifold or, more generally, a path metric space in the sense of [G]. A natural idea is to identify two paths γ_1 and γ_2 from $\Omega_x X$ having a common endpoint if the loop $\gamma_1 * \gamma_2^{-1}$ is not merely contractible but can be contracted via loops based at x of a controlled length. Of course, one needs to restrict the lengths of γ_1 and γ_2 to make this idea work. So, one needs to choose a positive value of a parameter U and to consider all paths of length $\leq U$ starting at the base point. Then one would like to identify two such paths γ_1 and γ_2 if $\gamma_1(1) = \gamma_2(1)$, and $\gamma_1 * \gamma_2^{-1}$ is contractible via loops of length $\leq V$ based at x, where $V \geq 2U$ is another parameter. If V is sufficiently large in comparison with U, then one just obtains the metric ball of radius U centered at x in the universal covering of X. However, in general, the introduced relation on the space of elements of $\Omega_x X$ of length $\leq U$ does not need to be transitive.

0.2. Yet we made the following observation: There exists a "controllably large" value of $V = V(U, s, x)$ that can be made larger than any prescribed number s such that the introduced relation is transitive and, therefore, an equivalence relation. As a result, the set of equivalence classes with respect to this relation covers X "away from the boundary". (Coverings away from the boundary will be rigorously introduced in Definition 1.1 below.) So, we obtain metric spaces $P(U, V)$ and their maps to X that have the same properties as the restriction of the universal covering map to metric balls in the universal covering (see Definition 1.B as well as Definition 1.1 and sections 1.3.1 - 1.3.4). However, in general, sets $P(U, V)$ are "larger" than the corresponding metric balls of radius U in the universal covering (as a weaker equivalence relation was used to define them). Also, suitable approximations to the sets $P(U, V)$ that also cover X away from the boundary can be constructed by means of an algorithm if X is presented in a finite form (see sections 1.2.3-1.2.5). (The last assertion is not true for metric balls in universal coverings of compact Riemannian manifolds-see section 1.2.2 for the details.) Therefore we call the metric spaces $P(U, V)$ effective universal coverings of X. Finally, note that the metric spaces $P(U, V)$ can be considered for simply-connected Riemannian manifolds. Moreover, $P(U, V)$ constructed for a closed simply-connected manifold M does not need to coincide with M even when U is large in comparison with the diameter of M (although when U will become very large $P(U, V)$ will eventually coincide with M; see section 2.5.2 for specific examples).

0.3. In this paper we will present two applications of this construction. We plan to present a third application, where the effectiveness of our construction plays an important role, in our forthcoming paper with Shmuel Weinberger [NW].

0.4. One application provides an effective version of a theorem proven by Michael Anderson ([A]) about compact n-dimensional Riemannian manifolds with Ricci curvature bounded from below by $-(n-1)$ (Theorem 1.7 below). (Note that hyperbolic manifolds have Ricci curvature $= -(n-1)$.) Assume that the diameter of such a manifold does not exceed d, and the volume is not less than a positive v. Anderson proved that there exist explicit $\delta(n, v, d) > 0, N(n, v, d)$ such that for every every closed curve γ of length $\leq \delta(n, v, d)$ one of the first $N(n, v, d)$ iterates of γ is contractible. (By the iterates of γ one means γ itself, γ^2 obtained by tracing γ twice, etc., $\gamma^{N(n,v,d)}$ obtained by tracing γ $N(n, v, d)$ times.) [1]

Now one can ask for a quantitative version of this result. Assume that we are given a closed loop γ based at a point x of length $\leq \delta(n, v, d)$. How large should be L to ensure that one of the first $N(n, v, d)$ iterates of γ is contractible via loops based at x of length $\leq L$? Note that this question is non-trivial even if the Riemannian manifold is simply-connected. We provide an answer for this question using our construction of "effective universal coverings". The original proof of Anderson uses a volume comparison argument for a metric ball in the universal covering of the Riemannian manifold. We follow the proof by Anderson but use our "effective universal coverings" instead of the universal covering. (The standard proof of the Bishop volume comparison estimate applies verbatim to bound the volumes of metric balls centered at the base point of the "effective universal covering" $P(U, V)$ and contained in the interior of $P(U, V)$ providing that the Ricci curvature of the underlying Riemannian manifold is bounded from below; see Theorem 1.3.4.)

0.5. Another application involves Morse landscapes of the length functional on a loop space of an arbitrary closed simply-connected Riemannian manifold.

Let M^n be a simply connected Riemannian manifold, x be a point of M^n, and $\Omega =$

[1] It seems that there is no positive δ depending only on n, v, d such that every closed curve of length $\leq \delta$ is always contractible, and one can construct examples that demonstrate this fact as a modification of an example constructed in the proof of Proposition 3.1 in [A]: One only needs to replace the Eguchi-Hanson metrics on TS^2 used in the proof of Proposition 3.1 in [A] by the analogous metrics on TRP^2. I learned of this idea from Vitali Kapovitch, and would like to thank him for telling me about this construction. I do not know if these (or other similar) examples have ever been published.

$\Omega_x M^n$ be the space of loops on M^n based at x. Ω is connected, and the length functional λ on Ω attains its global minimum (with value zero) at the trivial (constant) loop. In principle, the length functional might have no other local minima, as it happens, for example, in the case of a sphere with a canonical round metric. But assume that it has a non-trivial local minimum γ_1. Of course, γ_1 is a geodesic loop based at x.

The depth of a minimum of λ is given precisely in Definition 2.0 below and refers to the difference between the length of a longest loop in an "optimal" path homotopy from the minimum to the trivial loop and the length of the minimum.

One of the main purposes of this paper is to prove the following theorem:

Theorem 0.1. (Imprecise version). Let M^n be a simply-connected closed Riemannian manifold, p a point of M^n. If the length functional λ has a "very deep" non-trivial local minimum on $\Omega_p M^n$, then it has "many" "deep" local minima on $\Omega_p M^n$.

A precise version of this theorem (Theorem 2.1) as well as some of its corollaries can be found in section 2. (The theorem will be also true for simply-connected locally path connected compact path metric spaces satisfying a stronger version of semi-local simply-connectedness- see Defintion 1.2 below.) For example, for every k it is true that if λ has a local minimum of depth $\geq (4k^2+2k)d$, where d denotes the diameter of M^n, then λ has at least k local minima of level $\leq 4kd$. Moreover, one can ensure that the value of the length functional λ at these k local minima do not exceed $2d$ for the first of them and $4id$, $i=2,\ldots,k$ for the remaining ones.

Informally speaking, this theorem implies that a compact simply connected Riemannian manifold M cannot look like it has a finite fundamental group. (However, it is well-known that M can look like it has an infinite fundamental group - see section 2.5.2 below.)

0.6. One can also ask if the ratio of the depth of local minimum of λ to d can be arbitrarily large. (In particular, can a local minimum of X be as deep as required in order to apply this theorem in a non-trivial way?) The answer for this question is positive even if $n=2$, and M^n is diffeomorphic to the two-dimensional sphere. See section 2.5 below for two different ideas that lead to construction of such examples.

0.7. The idea of the proof of Theorem 0.1 (or rather its rigorous version Theorem 2.1) can be roughly explained as follows. A counterexample must look like it has a "small" finite fundamental group. (Otherwise, we will be able to construct many "deep" local minima of λ by taking powers and products of powers of already known loops providing local minima for λ and shortening them to geodesic loops providing (new) local minima.) Observe that when one constructs the usual universal covering of a compact Riemannian manifold with a finite fundamental group, then one does not need to consider arbitrarily long paths starting at x. Paths of length $\leq 2d \times |\pi_1(M^n)|$ are sufficient. (The longer paths turn out to be equivalent to shorter paths.) A similar phenomenon happens for "effective universal coverings": We prove that there exists U linearly dependent on k and d such that for a not very large value of V $P(U,V)$ will "close" into a closed manifold (i.e. the boundary of the "effective universal covering" $P(U,V)$ disappears). So, $P(U,V)$ will be covering M^n in the usual sense. This covering will be non-trivial since the "very deep" local minimum of λ and the trivial loop are two distinct points of $P(U,V)$ covering

x. Yet a simply-connected space cannot have a non-trivial covering, and we will obtain a contradiction.

1. Effective universal coverings.

1.1. Construction of effective universal coverings. Let X be a compact Riemannian manifold. Recall that in order to construct the universal covering space of X one proceeds as follows: Fix a point $x \in X$. Consider all paths starting at x. Two paths γ_1, γ_2 are equivalent if 1) they have a common endpoint; and 2) The loop $\gamma_1 * \gamma_2^{-1}$ obtained by going along γ_1 and then returning back via γ_2 is contractible. The universal covering space $\tilde{X}$ of X is then defined as the space of equivalence classes of the paths with respect to this relation. The map $\tilde{X} \longrightarrow X$ that sends each path starting at x to its endpoint is a covering map.

The following natural idea occured to the author many years ago when he first learned the construction of the universal covering space: What happens if one takes into consideration not only the contractibility/non-contractibility of $\gamma_1 * \gamma_2^{-1}$ but also the length of loops required to contract this loop? (The author is sure that this idea had occured also to many other mathematicians.) [2]

1.1.1. To be more precise let us fix positive numbers $U \geq d =$ diameter of X, $V \geq 2U$ and a point $x \in X$. Consider the set $P(U)$ of all paths of length $\leq U$ in X starting at x. A natural idea is to say that two such paths γ_1, γ_2 are equivalent if they have a common endpoint, and the loop $\gamma_1 * \gamma_2^{-1}$ can be contracted to x via loops of length $\leq V$ based at x. However, in general, this relation is not transitive, and therefore is not an equivalence relation. Indeed, assume that γ_i, $i = 1, 2, 3$ are three paths starting at x of length $\leq U$ that have a common endpoint. Assume that $\gamma_1 * \gamma_2^{-1}$ and $\gamma_2 * \gamma_3^{-1}$ each can be contracted to a point via loops of length $\leq V$. This does not mean that the same is true for $\gamma_1 * \gamma_3^{-1}$. The only obvious way of contracting $\gamma_1 * \gamma_3^{-1}$ that involves first inserting $\gamma_2^{-1} * \gamma_2$ and then contracting $\gamma_1 * \gamma_2^{-1}$ and $\gamma_2 * \gamma_3^{-1}$ leads to the upper bound $V + 2U$ for the length of loops in the contracting homotopy.

1.1.2. However, there is the following seemingly simple way around this difficulty: Note that $\gamma_1 * \gamma_2^{-1}$ is contractible if and only if there exists a path homotopy between γ_1 and γ_2. (Recall that a homotopy of paths with common endpoints fixing the endpoints is called a path homotopy.) This suggests the following way of defining an equivalence relation on the considered space of paths $P(U)$:

Definition 1.A. *For a given $V \geq U$ we say that γ_1 is V-equivalent to γ_2 if these paths have the same endpoints, and there exists a path homotopy between these paths that passes only through paths of length $\leq V$.*

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[2] In particular, I have also heard a similar idea from Shmuel Weinberger. This idea is also strongly suggested by the definition of local fundamental pseudogroups in section 5.6 of [G0]. (A more detailed exposition of results of [G0] can be found in [BK] or in section 7 of [Fu].) Yet note that in [G0] Gromov dealt only with the situation, when the lengths of the loops are less than $\frac{\pi}{2\sqrt{K}}$, where K is the supremum of the absolute value of the sectional curvature of X, and this constraint seems to be crucial for his approach (see section 2 of [BK] for more details).

It is easy to see that now we have an equivalence relation on $P(U)$, that we call V-equivalence.

Definition 1.B. Let $P(U)$ be the space of paths $\gamma : [0,1] \longrightarrow X$ such that $\gamma(0)$ is the base point x. We assume that $P(U)$ is endowed with C^0 topology. Define $P(U,V)$ as the space of V-equivalence classes of $P(U)$ endowed with the quotient topology. Define the covering map $\psi : P(U,V) \longrightarrow X$ by the formula $\psi([\gamma]) = \gamma(1)$. (Here $[\gamma]$ denotes the V-equivalence class of γ; it is obvious that $\gamma(1)$ does not depend on a choice of a particular representative in $[\gamma]$.)

In other words, the map $\psi : P(U,V) \longrightarrow X$ assigns to each V-equivalence class the common endpoint of all paths from this class. (Recall that we regard x as the starting point of all paths from $P(U)$. Now we are talking about the other endpoint.)

1.1.3. It is obvious that $P(U,V)$ is connected, ψ is surjective, and, at least, if $V > U$, then the inverse image of every point of X under ψ is a discrete set (as every two very close paths with the common endpoints can be connected by a path homotopy with almost no increase of the length).

1.1.4. A natural expectation is that ψ will be a covering map "away from the boundary". Before giving a rigorous definition of what we mean by a covering map away from the boundary in this context, we would like to note that for every $y \in P(U,V)$ the set S_y of all elements of $\psi^{-1}(y) \in P(U,V)$ is finite. To see this first note that every path from $P(U)$ is V-equivalent to its reparametrization proportionally to the arclength. Second, note the Ascoli-Arzela theorem implies the compactness of the set $\bar{P}(U)$ of all paths from $P(U)$ parametrized proportionally to the arclength. Finally, note that S_y is discrete (as it was explained in section 1.1.3).

Definition 1.1. We say that $\psi : P(U,V) \longrightarrow X$ is *a covering map away from the boundary* if there exists an ϵ_0 such that for every point $y \in X$ the following conditions hold. Let $y_1, \ldots, y_N$ be the set of all elements of $\psi^{-1}(y) \in P(U,V)$ that can be represented by a path of length strictly less than U connecting x and y. Denote the minimal length of a path representing y_i by l_i. Then for every positive $\epsilon \leq \frac{1}{2}\min\{\epsilon_0, U - \max_i\ l_i\}$ there exist disjoint open neighborhoods U_i of y_i in $P(U,V)$ such that: 1) The restriction of ψ on every set U_i is a homeomorphism of U_i and the open ball B_ϵ of radius ϵ centered at y in X; 2) $\bigcup_{i=1}^N U_i = \psi^{-1}(B_\epsilon) \bigcap P(U-\epsilon, V)$.

1.1.5. However, it is not always true that ψ is a covering away from the boundary. The reason for that can be explained as follows: It can happen that two paths γ_1, γ_2 of length $\leq U$ connecting x with a point y are V-equivalent but there exist arbitrarily short paths ν starting at y such that $\gamma_1 * \nu$ is not V-equivalent to $\gamma_2 * \nu$. If this happens, then ψ will not be a local homeomorphism in any neighborood of the V-equivalence class of γ_1 and γ_2.

1.1.6. Let us now return to the original idea of constructing effective universal covering that was outlined in section 1.1.1. We start from the following definition.

Definition 1.C. We say that two paths γ_1 and γ_2 of length $\leq U$ starting from x are V*-similar* if they have a common endpoint, and the loop $\gamma_1 * \gamma_2^{-1}$ can be contracted to x via loops of length $\leq V + U$ based at x.

1.1.7. Our first observation is that V-similarity becomes transitive and, thus, an equivalence relation providing that $V \geq 3U$ satisfies the following condition:

Condition (*): If a loop of length $\leq 2U$ based at x can be contracted to x via loops of length $V+U$ based at x, then it can be contracted to x via loops of length $\leq V-U$ based at x.

So, the idea of constructing of effective universal coverings started in section 1.1.1 but seemingly blocked by the lack of transitivity of V-smilarity will work if V satisfies condition (*).

Also, as we will see below in section 1.1.11 , if V satisfies condition (*), then our second approach to constructing effective universal coverings started in 1.1.2 can be carried through past the difficulty explained in section 1.1.5. To demonstrate this fact we will need the following lemma:

Lemma 1.1.7. Assume that $V \geq 3U$ satidfies Condition (*). Then the binary relations of V-equivalence (see Definition 1.A) and V-similarilty (see Definition 1.C) on $P(U)$ coincide.

Proof. Note, that if $V \geq 3U$ satisfies condition (*), then the notions of V-similarity and V-equivalence coincide. Indeed, if γ_1 and γ_2 can be connected by a path homotopy γ_t, $t \in [1,2]$, passing through paths of length $\leq V$, then $\gamma_1 * \gamma_2^{-1}$ can be contracted by, first, going through $\gamma_t * \gamma_2^{-1}$ to $\gamma_2 * \gamma_2^{-1}$, and then contracting $\gamma_2 * \gamma_2^{-1}$ over itself. The length of loops in this path homotopy does not exceed $V+U$. Conversely, if $\gamma_1 * \gamma_2^{-1}$ can be contracted by a path homotopy through loops of length $V+U$, then (*) implies that it can be contracted by a path homotopy H_t passing through loops of length $V-U$. Now γ_2 can be connected with γ_1 by the following path homotopy passing through paths of length $\leq V$: Start from $\gamma_2 = x*\gamma_2 = H_1*\gamma_2$. Go through paths $H_{1-t}*\gamma_2$ to $H_0*\gamma_2 = \gamma_1*\gamma_2^{-1}*\gamma_2$. Now cancel $\gamma_2^{-1} * \gamma_2$ over itself. □

1.1.8. Our second observation is that (*) follows from its particular case, where one considers not all contractible loops of length $\leq 2U$ based at x, but only contractible geodesic loops providing local minima of the length functional. More precisely:

Lemma 1.1.8. Assume that each geodesic loop of length $\leq 2U$ based at x and providing a local minimum for the length functional on $\Omega_x X$ that can be contracted to x via loops of length $\leq V+U$ based at x can also be contracted to x via loops of length $\leq V-U$ based at x. Then V satisfies condition (*).

Proof. Indeed, every loop α can be connected by a length non-increasing homotopy with a geodesic loop γ providing a local minimum for the length functional on $\Omega_p X$. (Informally, one can think here about something like a gradient flow for the length functional on $\Omega_x X$. Formally, one can apply the Birkhoff curve-shortening flow (cf. [C]. Note that [C] describes the Birkhoff curve-shortening flow for the space of closed curves. One can easily adapt it to the space of loops based at x by demanding that the endpoints are fixed at x.)) Now note that α cab be contracted to x by a homotopy passing via loops of length $\leq V+U$ (respectively, $V-U$) based at x if and only if γ can be contracted to x by a homotopy passing via loops of length $\leq V+U$ (respectively, $V-U$) based at x. □

1.1.9. Our next observation is that one needs to consider only one geodesic loop from each $(V-U)$-equivalence class of geodesic loops of length $\leq 2U$ based at x in order to verify (or ensure) that V satisfies condition (*). Now, if $V > 3U$, the Ascoli-Arzela theorem implies that one needs to ensure (*) for only finitely many non-trivial geodesic loops.

1.1.10. In this section we will use the observation made in sections 1.1.8, 1.1.9 to demonstrate that for every s there exists a "not very large" value of $V \geq s$ depending on U, X and x that satisfies condition (*) and to obtain specific upper bounds for such a value of V.

Choose any $s \geq 2U$. Let $N(U,s)+1$ denote the number of the s-equivalence classes of contractible geodesic loops of length $\leq 2U$ based at x and providing the local minima for the length functional on $\Omega_x X$. (Of course, this number is equal to the number of the s-equivalence classes of all contractible loops of length $\leq 2U$ based at x.) Note, that if $s > 2U$, then $N = N(U,s)$ is finite as an immediate corollary of the Ascoli-Arzela theorem and the obvious fact that very close paths with common endpoints can be connected by a homotopy fixing the endpoints with only a very small increase of length during the homotopy. (However, observe, that, if $s = 2U$ and X is not assumed to be an analytic Riemannian manifold, then $N(s,U)$ can, in principle, be infinite.) One of these $N+1$ s-equivalence classes is the equivalence class of the trivial loop; the remaining $N = N(U,s)$ classes contain only non-trivial loops.

Choose a representative γ_i from each of the remaining N s-equivalence classes. For every $i = 1, \ldots N$ let $t(\gamma_i)$ denote the minimal T such that γ_i can be contracted to a point via loops of length $\leq T$ based at x. Our next observation is that we need only to find an interval $(\alpha - U, \alpha + U]$ of length $2U$ free of the numbers $t(\gamma_i)$, where $\alpha \geq s + U$ in order for $V = \alpha$ to satisfy (*).

Finally, notice that the pigeonhole principle implies that one of the $N+1$ intervals $(s, s+2U]$, $(s+2U, s+4U]$, $\ldots$, $(s+2NU, s+(2N+2)U]$ has the desired property. Let j be the number of this interval. Then $V = s+(2j-1)U \leq s+(2N+1)U$ satisfies condition (*).

If we do not care about the value of s, and would like to have the value of V as small as possible, then we can have

$$V \leq \inf_{s \geq 2U} (s + (2N(U,s)+1)U), \qquad (1.1.1)$$

where $N(U,s) \geq 0$ denotes the number of non-trivial s-equivalence classes of geodesic loops of length $\leq 2U$ based at x. In particular, we can ensure that

$$V \leq (2N(U,2U)+3)U. \qquad (1.1.2)$$

If desired, we can improve the upper bound for V as follows: Let $G(U,s)$ denote the maximal number of geodesic loops $\gamma_i, i = 1, 2, \ldots, G(U,s)$ based at x of length $\leq 2U$ such that for every i $t(\gamma_i) \in (s+2(i-1)U, s+2iU]$. This definition of $G(U,s)$ implies that we can choose $V \in [s+U, s+(2G(U,s)+1)U]$. If we do not care about a lower bound for V, then we can choose

$$V \leq \inf_{s \geq 2U} (s + (2G(U,s)+1)U). \qquad (1.1.3)$$

In particular, we can ensure that

$$V \leq (2G(U, 2U) + 3)U. \qquad (1.1.4)$$

This assertion is obvious from the description of the procedure that we used to choose a value of V.

We summarize the discussion in this section by the following lemma:

Lemma 1.1.10. For every $s \geq 2U$ there exists V such that (a) V satisfies condition (*); (b) $V \geq s + U$; (c) $V \leq s + (2N(U, s) + 1)U$, and $V \leq s + (2G(U, s) + 1)U$. If condition (b) is replaced by a weaker condition $V \geq 3U$, then one can replace condition (c) by any of the inequalities (1.1.1)-(1.1.4).

1.1.11. Assume that $V > 3U$ satifies condition (*). Then Lemma 1.1.7 implies that the technical problem mentioned in section 1.1.5 that was preventing $\psi : P(U, V) \longrightarrow X$ from being a covering map away from the boundary (in the sense of Definition 1.1) disappears. Indeed, let γ_1, γ_2 be two V-equivalent paths, τ is a path starting at the common endpoint of γ_1 and γ_2 such that the lengths of $\gamma_1 * \tau$ and $\gamma_2 * \tau$ do not exceed U. Then γ_1 and γ_2 are V-similar. As a corollary, $\gamma_1 * \tau$ and $\gamma_2 * \tau$ are V-similar. (One can cancel $\tau * \tau^{-1}$ in the loop $\gamma_1 * \tau * \tau^{-1} * \gamma_2^{-1}$ over itself.) Now the fact that V satisfies condition (*) implies that V-similarity of $\gamma_1 * \tau$ and $\gamma_2 * \tau$ implies their V-equivalence.

1.1.12. Now assume first that X is a Riemannian manifold. It is obvious that the standard proof that the classical construction of the universal covering yields the universal covering can be adapted without any major changes to demonstrate that the map of $P(U, V)$ into X assigning to each path its endpoint is, indeed, the covering away from the boundary (in the sense of Definition 1.1). Once can choose ϵ_0 in the text of Definition 1.1 to be equal to the injectivity radius of M^n.

1.1.13. Now assume that X is not necessarily a Riemannian manifold but a path metric space in the sense of [G]. This means that X is a complete metric space such that the distance between every pair of points is equal to the length of a shortest path connecting these points in X. The standard construction of universal coverings can be performed on metric spaces that are connected, locally path connected and semi-locally simply connected (cf. [M]). The last requirement means that all closed curves in sufficiently small open sets are contractible. We need a slightly stronger constraint for our purposes. Namely, we need to require that all sufficiently short curves are contractible to a point via short curves. However, we allow some increase of the length during a contracting homotopy. More formally:

Definition 1.2. We will say that X is *strongly semi-locally simply connected* if there exists a positive ϵ_0 and a function $f : [0, \epsilon_0] \longrightarrow (0, \infty)$ such that (a) $\lim_{\epsilon \longrightarrow 0} f(\epsilon) = 0$; and (b) For every point $p \in X$, and $\epsilon \in (0, \epsilon_0]$ every two paths of length $\leq \epsilon$ starting at p and having a common endpoint can be connected by a path homotopy via paths of length $\leq f(\epsilon)$. (Equivalently, we could demand contractability of all loops of length $l \leq \delta_0$ for a small δ_0 via a path homotopy that passes through loops of length $\leq g(l)$, where g is a function defined for all positive $l \geq \delta_0$ and satisfying $g(l) \longrightarrow 0$, as $l \longrightarrow 0$.)

If X is path connected, locally path connected and strongly semi-locally simply connected, then the proof that $\psi : P(U, V) \longrightarrow X$ is a covering map away from the boundary for Riemannian manifolds generalizes for X without any technical difficulties.

To be a little bit more specific note that the only difference between the case when X is a Riemannian manifold and the more general case considered in this section is the following: When one constructs the universal covering of X and wants to construct an open neighborhood T of a lift of a point $p \in X$ represented by a path γ from x to p that is homeomorphic to a "small" open neighborhood Z of p in X, one first chooses a family of paths τ_q connecting p with all points $q \in Z$ in Z. Then one constructs T as the set of equivalence classes of paths $\gamma * \tau_q$ for all $q \in Z$. If X is a Riemannian manifold, and Z is contained in a ball of radius smaller than the injectivity radius $inj_p M^n$, then one can just choose the (continuous) family of (unique) minimizing geodesics between p and q as τ_q. In general, there is no unique or continuous choice of the family τ_q. However, this does not matter in the semi-locally simply-connected case, because all choices of paths connecting p and q in Z are equivalent (as the loop formed by any two such paths is contractible). In our situation we are not satisfied with these loops being merely contractible, but need them to be contractible via "negligibly" short loops, when Z is choosen to be sufficiently small.

Summarizing the previous discussion we obtain the following theorem:

Theorem 1.3. Let X be a compact Riemannian manifold or, more generally, a connected locally path connected strongly semi-locally simply connected path metric space. Let d denote the diameter of X, and x be a point of X. Choose any $U \geq d$ and $s \geq 2U$. Denote the number of s-equivalence classes of geodesic loops of length $\leq 2U$ based at x by $N+1$. There exists $V \in [s+U, s+(2N+1)U]$ that satisfies condition (*) above. Moreover, we can find a value of V satisfying (*) such that it satisfies any of the formulae (1.1.1)-(1.1.4). For any V satisfying condition (*) the map $\psi : P(U,V) \longrightarrow X$ is a covering away from boundary in the sense of Definition 1.1. (Recall that $P(U,V)$ and ψ were defined in section 1.1.2, Definition 1.B.)

Remark 1.4. Note that $P(U,\infty)$ is just the metric ball of radius U in the universal covering space of X constructed using paths starting at x. The quotient map $P(U,V) \longrightarrow P(U,\infty)$ is also a covering away from the boundary. The proof of this fact is straightforward.

Remark 1.5. Observe that condition (*) depends on U. But if V satisfies condition (*) for some U, it will satisfy (*) for all $U_* < U$. Moreover, the value of V that we found in this section increases with U (for a fixed value of s) and with s.

Remark 1.6. Observe that two paths of length $\leq U$ starting from x that are not $s+(2N+1)U$-equivalent correspond to different points of $P(U,V)$ for a value of V chosen as in the proof of Theorem 1.3.

We would like to finish section 1.1 with the following convention: Below in this paper the term *effective universal covering* will usually denote $P(U,V)$ for an appropriate value of V satisfying either one of the inequalities in condition (c) in Lemma 1.1.10 or one of the inequalities (1.1.1)-(1.1.4) providing that V satisfies condition (*) (or, equivalently, ψ is a covering away from the boundary in the sense of Definition 1.1.) Yet sometimes we will use this term to denote a subset of such a $P(U,V)$ that contains $P(U-\epsilon,V)$ for a small positive value of ϵ so that the restriction of ψ on this subset still satisfies all the conditions of Definition 1.1 (and is surjective).

1.2. Effectiveness of effective universal coverings.

1.2.1. It is a well-known fact (independently proven by S. Adyan and M. Rabin) that there is no algorithm deciding whether or not a given finite presentation of a group is a finite presentation of the trivial group (cf. a survey paper [Mi] or the original papers [Ad], [R]). As a consequence, there is no algorithm that decides whether or not a given compact manifold is simply-connected or not. (Here we assume that compact manifolds are presented by finite sets of data, e.g. by their triangulations, or as sets of solutions of a system of polynomial equations with algebraic coefficients in a Euclidean space of a higher dimension. See [BHP] for more details and a discussion of other algorithmic problems in topology of manifolds.)

1.2.2. This fact immediately implies that not only are there no algorithms constructing a good approximation to metric balls in the canonically constructed universal covering space of a Riemannian manifold, but there are no algorithms deciding many basic questions about these metric balls. Indeed, consider a metric ball of radius $3d$ centered at the base point in the canonically constructed universal covering space of a compact Riemannian manifold M, where d is the diameter of M. If M is simply-connected, then this metric ball coincides with M. Otherwise, it covers M at least twice, and if M has an infinite fundamental group, it has a boundary and diameter $\geq 3d$. So, there are no algorithms determining the most basic features of the geometry of this metric ball, or the restriction of the covering map to this ball.

1.2.3. However, here we provide an algorithm that for every given value of U, a given path metric space X satisfying conditions of Theorem 1.3 and presented by a finite set of data, and a point $x \in X$ constructs an effective universal covering of X. The algorithm first finds a value of V satisfying (*) and then, for any given positive ϵ, finds a subset of $P(U, V)$ that contains $P(U - \epsilon, V)$ and is a covering of X away from the boundary.

The constructed effective universal covering will also be a covering away from the boundary of the metric ball of radius U in the universal covering space of X (that can be identified with $P(U, \infty)$. The corresponding map is merely the restriction of the quotient map $P(U, V) \longrightarrow P(U, \infty)$ to the considered subset of $P(U, V)$.)

As we already mentioned, a path metric space X should be presented in a finite form. For example, we can assume that X is constructed from flat simplices (or polytopal faces), or that X is a Riemannian manifold presented by a semi-algebraic atlas and a set of semi-algebraic functions g_{ij} describing the Riemannian metric in local coordinates. (All semi-algebraic functions are assumed to be over the field of algebraic numbers. We also assume a high degree of smoothness of all transition functions and of the Riemannian metric.) Alternatively, we can assume that X is given as a semi-algebraic submanifold of an Euclidean space with the induced Riemannian metric. (Again, all semi-algebraic functions should be defined over the field of algebraic numbers.)

1.2.4. Indeed, note that the procedure for choosing a value of V that satisfies (*) described in section 1.1 can be modified as follows: First, note that an immediate computational difficulty is that we cannot find lengths of curves in X exactly, but only approximately. We start from constructing a dense net in the space of all loops based at x of length $\leq 2U$ parametrized by the arclength. Note that every pair of sufficiently close loops can be connected by a path homotopy that increases the length only by a small amount.

Assuming that $s > 2U$, we can ensure that for a sufficiently dense finite net in the space of loops these length increases do not exceed $s - 2U$. As we are unable to compute the length of curves exactly, we also include curves of length $\leq U + \epsilon$, where ϵ is a small parameter. If we are in doubt whether two closed curves are s-equivalent or only $s + \lambda$-equivalent for some small λ, we include them both to the collection of N representatives γ_i selected in the proof of Lemma 1.1.10. As the result, we will get not N but $N_* \geq N$ representatives. However N_* still does not exceed the number of $(s-\epsilon)$-equivalence classes of geodesic loops of length $2(U + \epsilon)$ based at x, where we can choose an arbitrarily small positive ϵ.

Next we modify the construction in section 1.1.10 (leading to the proof of the lemma at the end of that section) by choosing $\delta > 0$ and replacing consecutive intervals $(s+2(j-1)U, s+2jU]$ by the intervals of length $2U + 2\delta$, namely, $(s+2(j-1)U+(2j-1)\delta, s+2jU+(2j+1)\delta]$. This modification has two purposes: First, if we make a small error in determination of $t(\gamma_j)$, and decide that $t(\gamma_j)$ is outside of one of these intervals in the situation when it is inside of this interval, but δ-close to one of its ends, the smaller interval of length $2U + \delta$ centered at the same midpoint will be still free of $t(\gamma_j)$. Second, not only the midpoint of this interval can be used as a value of V satisfying (*), but every value $\delta/2$-close to the midpoint can be used as the value of V satisfying (*). In fact, V-equivalence of paths of length $\leq U$ will be exactly the same relation for all values of V in this interval of length δ. As the result, our algorithm becomes immune to small errors in determination whether or not two paths are V-equivalent. (If V-equivalence and $(V + \lambda)$-equivalence are the same equivalence relation, then the problem obviously disappears).

1.2.5. Once the value of V is chosen, we can construct a covering away from the boundary of X that is contained in $P(U, V)$ and contains $P(U-\epsilon, V)$ for an arbitrarily small ϵ (which should be a parameter of the algorithm). Here is a sketch of the construction. One first triangulates X into very small simplices (of diameter $\leq \epsilon/10$), such that each of these simplices σ_i is star-shaped with respect to an interior point q_i. Compute the set of all V-equivalence classes of paths between x and q_i of computed length $\leq U - \epsilon/2$, where the error of computation does not exceed $\epsilon/4$. Denote the number of these classes by N_i. Take N_i copies of the closed simplex σ_i. Each of these copies corresponds to the V-equivalence class of a geodesic γ_{il} between x and q_i. We can assume that each copy is marked by the corresponding V-equivalence class. We are going to glue the desired covering out of all these $\Sigma_i N_i$ simplices. We glue copies of simplices that are adjacent in X. Let $r \in \sigma_i \bigcap \sigma_j$. Take the lth copy of σ_i, $l \in \{1, \ldots, N_i\}$. We need to glue the point r in this copy to a point r in some copy of σ_j. To determine which copy to take we take $\gamma_{il} * [q_i r] * [r q_j]$ and determine its V-equivalence class. (Here $[q_i r]$ and $[r q_j]$ denote the minimal geodesics.) The V-equivalence class will determine the required copy of σ_j. It is important here that if r belongs to more than two of the considered simplices of the triangulation of x, then the "gluing" relation on the set of the simplices containing r is transitive. This transitivity follows from condition (*) of V.

1.3. Fundamental domains and an effective version of a theorem by M. Anderson. It is well-known (cf. [Pe]) that the universal covering of a compact Riemannian manifold M endowed with the pullback Riemannian metric can be tiled by fundamental domains. These domains admit the following description: If x_i is a lift of the base point x to the universal covering, then the interior of the fundamental domain containing x_i

consists of all points y such that $dist(x_i, y) < dist(x_j, y)$ for every other lift x_j of x to the universal covering. This set is called the Voronnoi cell of x_i. All these Voronnoi cells are isometric to the complement of the cut-locus of x in M.

In this section we first demonstrate that a "significant part" of an effective universal covering of M can be tiled by such domains. Then we observe that this picture can be used to estimate from above the number of V-equivalence classes of geodesic loops of length $\leq U - d$. This part of section 1.3 can be generalized to more general path metric spaces, e.g. those that are made out of flat or Riemannian simplices or polytopes.

Starting from section 1.3.4 we discuss a more special situation, when M is a Riemannian manifold with Ricci curvature bounded from below by $-(n-1)$. We observe that the interior of the effective universal covering $P_0(U, V)$ endowed with the pullback Riemannian metric satisfies the Bishop volume comparison inequality (Theorem 1.3.4). Then we discuss upper bounds for the number of V-equivalence classes of geodesic loops of length $\leq 2d$. Finally, we prove an effective version of a theorem by M. Anderson (Theorem 1.7).

1.3.1. Let M be a compact n-dimensional Riemannian manifold of diameter d. Assume that V satisfies condition (*). Then one can tile a subset of $P(U, V)$ containing $P(U-d, V)$ by copies a connected fundamental domain, which is constructed exactly as in the case of "usual" universal coverings. The proof is almost identical to the proof of this fact for "usual" universal coverings:

Consider the complement S to the cut-locus of x. Take the inverse image F of S under the exponential map $\exp_x$. The exponential map is a diffeomorphism between F and S and maps the closure of F surjectively onto M. If we endow F with the Riemannian metric which is the pullback of the Riemannian metric on M by the exponential map, then the exponential map becomes an isometry between F and S. We can embed F into $P(d) \subset P(U)$ by identifying each point f of F with the corresponding minimal geodesic $\exp_x([0f])$ in M. As all these geodesics have different endpoints, F embeds into $P(U, V)$ for every value of V.

We would like to verify that each path γ of length $\leq U - 2d$ starting at x and ending at a point of S is V-equivalent to the join of a geodesic loop λ of length $\leq U - d$ based at x and a minimal geodesic between x and the endpoint of γ. Indeed, attach to γ the minimal geodesic between its endpoint and x and then apply the Birkhoff curve-shortening flow with fixed endpoints to the resulting loop. This flow stops at a (possibly trivial) geodesic loop that we will denote λ. The original path γ is V-equivalent (and even U-equivalent) to $\lambda * [x\gamma(1)]$, where $[\gamma(1)x]$ denotes the minimal geodesic that we have used in this construction.

Note that $\lambda_1 * [xr]$ is V-equivalent to $\lambda_2 * [xr]$ if and only if λ_1 and λ_2 are V -equivalent. (Indeed, our choice of V implies that V-equivalence is the same property as V-similarity, and this assertion for V-similarity instead of V-equivalence is obvious.)

Let now μ be a V-equivalence class of loops of length $\leq U - d$ based at x. Then V-equivalence classes of paths $\mu * [xr]$, where r runs over S form a subset P_μ of $P(U, V)$ homeomorphic to F. Moreover, as $dist(x, r) < d$ for each $r \in S$ the length of $\mu * [xr]$ is strictly less than U. Denote the subset of $P(U, V)$ formed by V-equivalence classes representable by paths of length $< U$ by $P_0(U, V)$. Definition 1.1 implies that ψ is a local homeomorphism in a neighbourhood of each point of $P_0(U, V)$. Therefore we can endow

$P_0(U,V)$ with a pullback Riemannian metric under the restriction of ψ on $P_0(U,V)$. Now $P_\mu \subset P_0(U,V)$ becomes not only homeomorphic but isometric to F.

Thus, $P(U-2d,V)\bigcap\psi^{-1}(S)$ is contained in the union of disjoint sets P_μ contained in $P_0(U,V)\subset P(U,V)$ and isometric to F, where μ runs over the sets of all $N(\frac{U-d}{2},V)+1$ V-equivalence classes of loops of length $\leq U-d$ based at x.

1.3.2. As all sets P_μ are isometric to $F=M\setminus S$, we conclude that $vol(P_\mu)=vol(F)=vol(M\setminus S)=vol(M)$. As all $N(\frac{U-d}{2},v)+1$ sets P_μ are contained in $P_0(U,V)$ we see that:

Lemma 1.3.2. *$vol(P_0(U,V))\geq v(N(\frac{U-d}{2},V)+1)$, where v denotes the volume of M.*

Note that the arguments in section 1.3.1 and 1.3.2 can be vastly generalized to a wide class of path metric spaces.

1.3.3. Here we would like to record some obvious properties of metric spaces $P_0(U,V)$ endowed with the pullback Riemannian metrics under the restriction of ψ to $P_0(U,V)$. (Here, as everywhere else in section 1.3, we are assuming that V satisfies condition (*).) Note that $P_0(U,V)$ is connected, as we can connect each $[\gamma]\in P_0(U,V)$ with the V-equivalence class of the trivial path $[x]$ via V-equivalence classes of subpaths of $[\gamma]$. The distance between $[x]$ and $[\gamma]$ is equal to the minimal length of a path between x and $\gamma(1)$ in the same V-equivalence class as γ. Therefore, $P_0(U,V)$ is the open metric ball of radius U centered at $[x]$ in itself. Note that $P_0(U,V)$ is the interior of $P(U,V)$, and the boundary of $P_0(U,V)$ consists of V-equivalence classes of paths of length U starting at x that are not V-equivalent to paths of a smaller length.

Further, $P_0(U,V)$ is locally isometric to M, and therefore satisfies the same curvature bounds as M. Also, it is clear that each geodesic in $P_0(U,V)$ can be extended until it will approach the boundary of $P_0(U,V)$. This implies that one can define the exponential map $\exp_{[x]}:B_U(0)\longrightarrow P_0(U,V)$, where $B_U(0)$ denotes the open ball of radius U centered at the origin of the n-dimensional tangent space $T_{[x]}P_0(U,V)$ to $P_0(U,V)$ at $[x]$. This map is surjective and has all the usual properties of an exponential map on a Riemannian manifold.

1.3.4. Assume now that the Ricci curvature of M is bounded from below by $-(n-1)$. Then the classical Bishop volume comparison inequality implies that the volume of a metric ball of radius r in M or in a covering of M does not exceed the volume $v_n(r)$ of a metric ball of radius r in the n-dimensional hyperbolic space. The classical proof of this assertion (cf. [Pe]) can be repeated verbatim to prove that the same upper bound will be true for for the volume of metric balls of radius $r<U$ centered at a point x in an open n-dimensional Riemannian manifold X with $Ric\geq -(n-1)$ providing that the exponential map $\exp_x:B_0(U)\longrightarrow X$ is defined on the open metric ball of radius U in T_xX centered at the origin. Indeed, no part of the proof of the original Bishop volume estimate for $vol(B(x,r))$ appeals to the part of the manifold outside of the closure of the metric ball $B(x,r)$ of radius r centered at x in X. Passing to the limit as $r\longrightarrow U$, we see that the same upper bound holds for $r=U$ as well.

Now taking into account that $P_0(U,V)$ has all the required for this argument properties (see section 1.3.3) we can conclude that:

Theorem 1.3.4. *$vol(P_0(U,V))\leq v_n(U)$.*

Remark. Note that this inequality is, in general, stronger than the corresponding inequality for the volume of metric balls $P(U,\infty)$ in the universal covering space of M.

1.3.5. In this section we are going to assume that V satisfies condition (*), and M is a Riemannian manifold with Ricci curvature $\geq -(n-1)$. The purpose of this section is to discuss the upper bounds for the number of V-equivalence classes of geodesic loops of a bounded length that follow from Lemma 1.3.2 and Theorem 1.3.4. We feel that this subject is of an independent interest, but it is also motivated by its application to the proof of Theorem 1.7 below.

Combining Theorem 1.3.4 with Lemma 1.3.2 we see that

$$N(\frac{U-d}{2},V)+1\leq \frac{v_n(U)}{v}. \qquad (1.3.5.1)$$

In particular, for $U=3d$ we obtain

$$N(d,V)+1\leq \frac{v_n(3d)}{v}. \qquad (1.3.5.2)$$

Note that the right hands of these inequalities do not depend on V. Thus, they become the strongest when V is the minimal possible. Thus, we can substitute the right hand sides of one of the inequalities (1.1.1)-(1.1.4) for V. (The inequality (1.1.3) provides the strongest bound.) But note that we do not know an answer for the following interesting

Question 1.6. Is there an upper bound for $N(d,V)$ in terms of n,d,v for smaller values of V? In particular, is there such a bound for $V=3d$, or at least for a value of V not exceeding a certain function of n,d and v?

1.3.6. The purpose of this section is to prove the following effective version of a theorem by M. Anderson:

Theorem 1.7. Let M^n be a compact Riemannian manifold with Ricci curvature $\geq -(n-1)$, volume $\geq v$ and diameter $\leq d$. Let $K=\lfloor\frac{v_n(3d)}{v}\rfloor$, $\delta=\frac{d}{K}$, where $v_n(3d)$ denotes the volume of a metric ball of radius $3d$ in the n-dimensional hyperbolic space. Let γ be a closed loop in M^n based at a point x of length $\leq\delta$. Then one of the first K iterates of γ can be contracted to x by a path homotopy that passes only through loops of length $\leq \inf_{r\geq 0}(r+2dN(d,2d+r))+(4-\frac{1}{K})d$.

Proof. Choose $K=\lfloor\frac{v_n(3d)}{v}\rfloor > N(d,V)$. (This inequality follows from the inequality (1.3.5.2).) Let $\delta=\frac{d}{K}$. Assume that there exists a geodesic loop γ of length $\leq\delta$ based at x. By the pigeonhole principle, either one of the first K iterates of γ is contractible via loops of length $\leq V$, or two of these iterates, say, γ^i and γ^j, $i>j$, are V-equivalent to each other. In the last case, γ^{i-j} can be contracted by first inserting $\gamma^j*\gamma^{-j}$, then homotoping $\gamma^{i-j}*\gamma^j=\gamma^i$ to γ^j and, finally, cancelling $\gamma^j*\gamma^{-j}$. It is easy to see that the resulting homotopy passes through loops based at x of length $\leq V+$ length$(\gamma^j)\leq V+(K-1)\delta=V+(1-\frac{1}{K})d$. According to (1.1.2) we can choose $V=(2N+3)d$, if desired, where $N=N(d,2d)$ denotes the number of non-trivial $2d$-equivalence classes of geodesic loops of length $\leq 2d$ based at x on M. Thus, at least one of the first K iterates of γ is $(2N+4-\frac{1}{K})d$-equivalent to the constant geodesic. Instead, we can use upper bounds for V provided by inequalities (1.1.1), (1.1.3), (1.1.4). Using the inequality (1.1.1) we obtain Theorem 1.7. ☐

Again, recall that only the upper bound for the length of loops in a contracting homotopy is new in Theorem 1.7. We can further improve the inequality in theorem 1.7 by replacing the upper bound (1.1.1) for V by the upper bound (1.1.3).

2. Morse landscape of the length functional on loop spaces.

2.1. Main results. We are going to start from the rigorous definition of the *level* and *depth* of a local minimum.

Definition 2.0. Let $\nu : X \longrightarrow R$ be a continuous functional defined on a path connected space X. Assume that ν attains a global minimum on X. For every local minimimum z of ν we define the *level* of z as the infimum of all h such that there exists a continuous path $\tau : [0,1] \longrightarrow X$ starting at z, passing through points where the value of ν does not exceed h and ending at a global minimum of ν. We will call the difference between the level of z and $\nu(z)$ the *depth* of z.

This definition has the following meaning: Assume that γ_1 is a local minimum of λ on Ω, and the level of γ_1 is h, Then there exists a path homotopy contracting γ_1 to a point that passes only through loops of length $\leq h$, but there is no such path homotopy passing only through loops of length $< h$. The depth of γ_1 measures how much one needs to increase the length of γ_1 before it becomes contractible to a point.

Now we are going to state a rigorous version of Theorem 0.1 from the introduction:

Theorem 2.1. Let M^n be a simply connected closed Riemannian manifold of diameter d, $x \in M^n$, $k > 1$ an integer, and $s \geq 4kd$ a real number. Assume that the length functional $\lambda : \Omega_x M^n \longrightarrow R$ has a non-trivial local minimum $\rho \in \Omega_x M^n$ of depth greater than $S = s + (4k^2 - 2k)d$. Then λ has at least k non-trivial local minima $\gamma_1, \ldots, \gamma_k$ of level greater than or equal to s such that:
1) length$(\gamma_1) \leq 2d$, length$(\gamma_i) \leq 4id$ for $i = 2, 3, \ldots, k$;
2) For every pair i, j such that $i \neq j$ there is no path homotopy between geodesics loops γ_i and γ_j passing through loops of length $\leq s$.

If $\lambda(\rho) \leq 2d$, then it is sufficient to assume that the level (not the depth!) of ρ does not exceed S.

Remark 2.1.1. It will be clear from the proof of Theorem 2.1 that Theorem 2.1 is true (with the same proof) in a more general case, when M^n is not assumed to be a closed n-dimensional Riemannian manifold, but an arbitrary compact simply-connected locally path connected strongly semi-locally simply connected path metric space. (Recall, that strong semi-local simply connectedness was introduced by Definition 1.2.)

Remark 2.1.2. The proof of Theorem 2.1 given below will produce the geodesic loops $\gamma_1, \ldots, \gamma_k$ of different lengths. In particular, they cannot differ from each other only by the reversal of the orientation.

Note that every loop is path homotopic to a geodesic loop providing a non-trivial local minimum of λ via a length-nonincreasing path homotopy. Indeed, one can just apply the

Birkhoff curve-shortening flow with fixed endpoints. (See [C] for a detailed description of the Birkhoff curve-shortening process on the space of closed curves. The situation when the base point of a loop is fixed during the curve-shortening process is completely similar.) Therefore, one can state Theorem 2.1 in the following equivalent form (which will be proven in the next section):

Theorem 2.1.A. Let M^n be a simply connected closed Riemannian manifold of diameter d, $x \in M^n$, $k > 1$ an integer, and $s \geq 4kd$ a real number. Assume that there exists a loop γ of length l based at x that cannot be contracted to a point by a path homotopy that passes through loops of length $\leq S + l$, where $S = s + (4k^2 - 2k)d$. If $l \leq 2d$, it is sufficient to assume that γ cannot be contracted to a point by a path homotopy passing through loops of length $\leq S$.

Then there exist k distinct non-trivial geodesic loops $\gamma_1, \ldots, \gamma_k$ based at x with the following properties:
1) All these loops are local minima of λ on Ω;
2) The length of γ_1 does not exceed $2d$; the length of γ_i does not exceed $4id$ for $i = 2, 3, \ldots k$;
3) There is no path homotopy contracting any of these k loops and passing through loops of length $\leq s$. Also, there is no path homotopy connecting any two of these loops with each other and passing through loops of length $\leq s$.

We can improve the constants in Theorem 2.1.A in the case, when $k = 2$ (which we regard as the first non-trivial case). [3] In particular, we prove:

Theorem 2.2. Let M^n be a simply connected Riemannian manifold of diameter d, $x \in M^n$. Assume that there exists a loop γ based at x which either has a length $\leq 2d$ and cannot be contracted to a point by a path homotopy via loops of length $\leq 12d$, or has an arbitrary length l and cannot be contracted via loops of length $\leq l + 12d$. Then there exist two distinct geodesic loops γ_1, γ_2 based at x such that:
1) length$(\gamma_1) \leq 2d$; length$(\gamma_2) \leq 6d$;
2) γ_1 cannot be contracted by a path homotopy passing through loops of length $\leq 12d$; γ_2 cannot be contracted by a path homotopy passing through loops of length $\leq 6d$; γ_2 cannot be connected by a path homotopy passing through loops of length $\leq 6d$ with γ_1 or γ_1^{-1};
3) γ_1 and γ_2 are local minima of the length functional on the space of loops based at x.

Theorem 2.2 will be proven in section 2.4. Taking $s = 4kd$ we obtain the following immediate corollary of Theorem 2.1.A:

Corollary 2.3. Let M^n be a simply connected closed Riemannian manifold of diameter d, and $k > 1$ be an integer. Let γ be a loop of length l based at a point $x \in M^n$ that cannot be contracted to a point by a path homotopy passing through loops based at x of length $\leq l + (4k^2 + 2k)d$. Then there exist at least k non-trivial geodesic loops γ_i based

[3] We regard the case $k = 1$ as trivial. Indeed, if $l \leq 2d$, then one can obtain γ_1 by applying the Birkhoff curve-shortening process on Ω to γ. If $l > 2d$, then one can obtain γ_1 using a simple trick explained in the proof of Lemma 2.5 in the next section.

at x such that the length of γ_1 does not exceed $2d$, and the length of γ_i, $(i = 2, \dots, k)$, does not exceed $4id$. These geodesic loops provide local minima for the length functional. Moreover, none of these k geodesic loops can be contracted to a point or connected with another of these geodesic loops by a path homotopy passing only through loops of length less than or equal to $4kd$.

Now let y be an arbitrary point of M^n. We can consider the length functional on the space $\Omega_{x,y}M^n$ of all paths starting from x and ending at y.

Corollary 2.4. Let M^n be a closed Riemannian manifold of diameter d. Let x, y be two points on M^n, k a positive integer number and $s \geq 4kd$ a real number. Assume that there exists a closed loop γ based at x of length l such that γ cannot be contracted by a path homotopy passing through loops of length $\leq l + s + (4k^2 - 2k)d$. Then there exist at least $k + 1$ distinct geodesics g_i between x and y such that g_1 is a shortest geodesic between x and y, the length of g_2 does not exceed $2d + dist(x, y)$, and for every $i = 3, 4 \dots, k + 1$ the length of g_i does not exceed $\leq 4kd + dist(x, y)$. The geodesics g_i are local minima for the length functional on $\Omega_{x,y}$. There are no path homotopies between these geodesics passing only through paths of length $\leq s - dist(x, y)$.

Proof. Indeed, Theorem 2.1 implies the existence of k non-trivial loops λ_i of length $\leq 4id$ such that no pair of them can be connected by a path homotopy that passes through loops of length $\leq s$. (The length of λ_1 does not exceed $2d$.) Fix a shortest geodesic τ between x and y. For every $i = 1, \dots, k$ consider the path $\rho_i = \lambda_i * \tau$ obtained by first going along λ_i and then along τ. For $i = 2, 3 \dots, k + 1$ let g_i denote a geodesic between x and y of the smallest possible length among all geodesics between x and y that can be connected with ρ_{i-1} by a path homotopy passing via paths of length $s - dist(x, y)$. (It is clear that $\text{length}(g_i) \leq \text{length}(\rho_{i-1})$.) Let $g_1 = \tau$. Then $g_i, i = 1, \dots, k + 1$ will be $k + 1$ geodesics satisfying the conclusion of Corollary 2.4. Indeed, assume the opposite. Then there exists $i, j, i < j$ such that ρ_i and ρ_j are path homotopic via paths of length $\leq s - dist(x, y)$. Denote this path homotopy by P_{ij}. Then λ_i will be homotopic to λ_j via loops based at x of length $\leq s$, and we obtain a contradiction. The path homotopy between λ_i and λ_j is constructed first by inserting $\tau * \tau^{-1}$, then by using P_{ij} to pass from $\lambda_i * \tau$ to $\lambda_j * \tau$, and finally by cancelling $\tau * \tau^{-1}$. □

To prove Theorem 2.1 we first reduce it to a version (Proposition 2.6), where constraints on the lengths of the k local minima γ_i of the length are not so strict. Then we proceed by contradiction. Consider the "effective universal coverings" of M^n. The basic idea of the proof is that if U is sufficiently large, then the space of the equivalence classes will "close" into a closed manifold, similarly to what it happens when one constructs the universal covering of a compact Riemannian manifold with a *finite* fundamental group. (One does not need to consider arbitrarily long paths to construct the universal covering in that case.) Then the covering away from the boundary will become a usual covering. This covering will be non-trivial as the deep local minimum ρ and the trivial loop will be two different points above x. Now we obtain a contradiction with the fact that M^n being simply-connected does not admit non-trivial coverings.

2.2. Length of locally minimizing loops in Theorem 2.1. In this section we present simple arguments that reduce Theorem 2.1 to its version where the constraints on lengths of k local minima of λ are weaker. (This version is stated below as Proposition 2.6.) We start from the following simple lemma:

Lemma 2.5. Let M^n be a closed Riemannian manifold of diameter d, x a point on M^n, and $S > 2d$ a real number. Let γ be a loop of length l based at x such that γ cannot be contracted to a point by a path homotopy passing via loops of length $l + S$. Then there exists a geodesic loop γ_1 of length $\leq 2d$ that cannot be contracted to a point via loops of length $\leq S$ based at x.

Proof. Take a small $\epsilon \in (0, S - 2d)$. Subdivide γ into arcs $[t_i t_{i+1}]$ of length $\leq \epsilon$, $(i = 1, \ldots, N)$. Connect $x = \gamma(t_1) = \gamma(t_N)$ with each of the points $\gamma(t_i)$ by a minimal geodesic μ_i. Consider triangles T_i formed by μ_i, $[t_i, t_{i+1}]$ and μ_{i+1}. The length of each of these triangles does not exceed $2d + \epsilon$. Apply the Birkhoff curve-shortening process to each of these triangles. Denote a (possibly trivial) geodesic loop obtained from T_i as the result of this process by σ_i. We claim that at least one of these geodesic loops is not trivial and cannot be contracted to a point by a path homotopy passing through loops of length $\leq S$. Indeed, otherwise we can contract γ as follows: Let ν_i be formed by μ_i and the arc of γ between t_i and $t_N = x$. In particular, $\nu_1 = \gamma$, ν_N is the constant loop based at x. Start from $\gamma = \nu_1$ and proceed by induction. At each step we will connect ν_i with ν_{i+1} by the following path homotopy. Using our assumption we can create σ_i by a path homotopy that passes through loops of length $\leq S$ based at x. Therefore we can connect ν_i and $\sigma_i^{-1} * \nu_i$ by a homotopy that passes through loops of length $\leq l + S$ based at x. Then we can use the length non-increasing homotopy between σ_i and T_i in order to pass from $\sigma_i^{-1} * \nu_i$ to $T_i^{-1} * \nu_i = \mu_{i+1} * [t_{i+1}, t_i] * \mu_i^{-1} * \mu_i * [t_i, t_{i+1}] * [t_{i+1}, t_N]$ via loops of length $\leq l + 2d + \epsilon \leq l + S$. Finally, we cancel $\mu_i^{-1} * \mu_i$ and, then, $[t_{i+1}, t_i] * [t_i, t_{i+1}]$ by length decreasing homotopies and obtain ν_{i+1}. Combining all these homotopies we can contract $\gamma = \nu_1$ to ν_N, which is the constant loop.

Thus, we proved that for every $\epsilon > 0$ there exists a geodesic loop of length $\leq 2d + \epsilon$ that cannot be contracted to a point via loops of length $\leq S$. Taking $\epsilon \longrightarrow 0$ and using a compactness argument we obtain a desired geodesic loop γ_1. This completes the proof of the lemma. □

Now we are going to state the following modification of Theorem 2.1:

Proposition 2.6. Let M^n be a simply connected closed Riemannian manifold of diameter d, s a positive real number and $k > 1$ an integer. Let γ be a loop of length l based at a point $x \in M^n$ that cannot be contracted to a point by a path homotopy passing through loops of length $\leq S$, where $S = \max\{2\max\{l-d, 0\} + (4k-2)d, s\} + (2k-1)\max\{l-d, 0\} + (4k^2-4k+1)d$. Then there exist k non-trivial geodesic loops of length $\leq 2\max\{l-d, 0\} + (4k-2)d$. In addition, no two of these k geodesic loops can be connected with each other or contracted to x by a path homotopy passing through loops of length $\leq s$.

Note that by combining Proposition 2.6 with Lemma 2.5 we obtain Theorem 2.1:

Proof of Theorem 2.1.A assuming the validity of Proposition 2.6. If $l \leq 2d$, apply the Birkhoff curve-shortening process with fixed endpoints to γ. It terminates at a geodesic loop based at x that cannot be contracted to a point via loops based at x of length $\leq S$ and, therefore, is non-trivial. (Here we are using the value of S defined in the text of Theorem 2.1.A.) Denote this loop by γ_1. If $l > 2d$ apply Lemma 2.5 to γ to obtain the geodesic loop γ_1. In both cases the geodesic loop γ_1 has length $\leq 2d$. It cannot be contracted to a point by a path homotopy passing through loops of length $\leq S$.

Now we are going to apply Proposition 2.6 to γ_1 instead of γ and to each $i = 2, 3, \ldots, k$ instead of k. As $\max\{\text{length}(\gamma_1) - d, 0\} \leq d$, the value of S in the text of Proposition 2.6 does not exceed the value of S in the text of Theorem 2.1.A. Therefore the assumptions of Theorem 2.1.A imply the assumptions of Proposition 2.6. At the end of the ith step we obtain i distinct geodesic loops $\gamma_1, \ldots, \gamma_i$ of length $\leq 4id$ that cannot be contracted to a point or connected with each other by a path homotopy passing through loops of length $\leq s$. The application of Proposition 2.6 on the ith step produces i geodesic loops based at x of length $\leq 4id$ that have distinct non-trivial s-equivalence classes. The s-equivalence classes of up to $(i-1)$ of these loops can be the same as the s-equivalence classes of the loops obtained at the previous step. But at least one of these i s-equivalence classes does not coincide with the s-equivalence classes of $\gamma_1, \ldots, \gamma_{i-1}$. We can take the corresponding loop to be γ_i. This will end the ith step of the construction. After performing all steps up to $i = k$, we will obtain the desired geodesic loops $\gamma_1, \ldots, \gamma_k$. This completes the proof of Theorem 2.1 modulo Proposition 2.6. □

So, it remains to prove Proposition 2.6 in order to prove Theorem 2.1.

2.3. Proof of Proposition 2.6. In this section we prove Proposition 2.6. The proof of this proposition is briefly described here and then given in detail using few lemmae below.

First, note that if γ has level $> S$, then it has level $> S + \epsilon$ for a small positive ϵ. Now let $s_1 = \max\{2\max\{l-d, 0\} + (4k-2)d, s\} + \epsilon$. We assume that γ is not $s_1 + (2k-1)\max\{l-d, 0\} + (4k^2 - 4k + 1)d$-equivalent to the trivial loop. We are going to prove that for an arbitrary small positive $\delta < \frac{\epsilon}{4k}$ there exist at least k loops based at x of length $\leq 2\max\{l-d, 0\} + (4k-2)d + 4k\delta$ that are pairwise not s_1-equivalent and not s_1-equivalent to the trivial loop. As the lower bound for the level of each of these loops is greater than their lengths and does not depend on δ, we can pass to the limit as $\delta \longrightarrow 0$. In the limit we will obtain k geodesic loops of length $\leq 2\max\{l-d, 0\} + (4k-2)d$, and these loops will still be pairwise not s_1-equivalent and not s_1-equivalent to the trivial loop.

Thus, we are going to prove the existence of k pairwise not s_1-equivalent loops of length $\leq 2\max\{l-d, 0\} + (4k-2)d + 4k\delta$ which are also not s_1-equivalent to the trivial loop. We are going to do this by contradiction. Assume that there are at most $k-1$ such loops. We are going to construct an effective covering $P(U, V)$ of M^n. We will demonstrate that $U \leq \max\{l-d, 0\} + (2k-1)d + 2k\delta$ can be chosen so that $P(U, V)$ becomes a closed manifold, and $\psi : P(U, V) \longrightarrow M^n$ becomes a covering in the usual sense. Moreover, U is sufficiently small to guarantee that one can choose $V \leq S$ here, which implies that this covering is not trivial, as γ and the trivial loop will be two distinct points of $P(U, V)$ mapped by ψ to x. As M^n is simply-connected, we obtain a contradiction with the well-known fact, that each covering of a simply-connected space is a homeomorphism.

2.3.1. Choice of U. Recall that we are assuming that there are at most $k-1$ non-trivial s_1-distinct geodesic loops based at x of length $\leq 2\max\{l-d,0\}+(4k-2)d+4k\delta$. Fix a system of "short" s_1-distinct (=pairwise not s_1-equivalent) geodesic loops based at x as follows: Denote the trivial loop based at x by g_1. Choose a shortest geodesic loop g_2 not s_1-equivalent g_1. Then for every $m=3,\ldots$ choose a shortest geodesic loop g_m not s_1-equivalent to the already choosen loops $g_1,\ldots,g_{m-1}$. The process stops once the length of g_m becomes greater than $\max\{l-d,0\}+(2k-1)d+2k\delta < 2\max\{l-d,0\}+(4k-2)d+4k\delta$. Obviously, this happens for $m\leq k+1$. (Otherwise our assumption about the number of geodesic loops is false.) Let $L=\max\{d,l\}$. Consider k disjoint closed intervals $[L+\delta-d, L+2\delta+d], [L+3\delta+d, L+4\delta+3d],\ldots,[L+(2k-1)\delta+(2k-3)d, L+2k\delta+(2k-1)d]$. For at least one of these intervals there are no geodesic loops $g_i, i=1,\ldots,m-1$ based at x such that the length of g_i lies in this interval. (The length of the trivial loop is not in one of these k intervals, and there are at most $k-1$ remaining geodesic loops g_i, $i=2,\ldots m-1$.) Assume that $I_j=[L+(2j-1)\delta+(2j-3)d, L+2j\delta+(2j-1)d]$ is one of such intervals. Then we are going to choose $U=L+2j\delta+2(j-1)d$. Observe that $U\leq L+(2k-2)d+2k\delta=\max\{l-d,0\}+(2k-1)d+2k\delta$.

The following lemma provides a motivation for our choice of U:

Lemma 2.7. Assume that U is as chosen in the prior paragraph. For every $V\geq\max\{U+2d, s_1+d\}$ every path τ of length $\leq U$ connecting x with a point $p\in M^n$ is V-equivalent to a path of length $\leq U-\delta$ connecting x with p.

Proof of Lemma 2.7. Let σ denote a minimizing geodesic from x to p, and μ be the loop formed first by following τ and then by returning back to x along σ. Let μ_* denote a shortest loop s_1-equivalent to μ. Then either μ_* coincides with one of the loops $g_1, g_2, g_3, \ldots g_{m-1}$, or it has exactly the same length as one of these loops. Therefore the length of μ_* is *not* in I_j. On the other hand it does not exceed $U+d$. Therefore it should be less than $\min I_j=U-d-\delta$.

Consider the following path homotopy of paths from x to p: Start with τ. Gradually attach to it larger and larger arcs of σ^{-1} together with the same arcs traversed in the opposite direction. Finally, we obtain $\tau*\sigma^{-1}*\sigma$. Now consider the path homotopy of loops that starts from $\tau*\sigma^{-1}=\mu$, ends at μ_* and passes through loops of length $\leq s_1$. Attach σ at the end of each of these loops. As a result, one obtains a path homotopy between $\tau*\sigma^{-1}*\sigma=\mu*\sigma$ and $\mu_**\sigma$ that passes through paths of length not exceeding s_1+d. The length of the path $\mu_**\sigma$ is less than $(U-d-\delta)+d=U-\delta$. This completes the proof of the lemma. □

2.3.2. The value of V. According to the inequality (1.1.1) one can choose a value of V not exceeding $s_1+(2N(U,s_1)+1)U$ that satisfies condition (*) from section 1.1 (and which is also not less than s_1+U). Note that $N(U,s_1)\leq N(\max\{l-d,0\}+(2k-1)d+2k\delta, s_1)$. The right hand side of this inequality is the number of non-trivial s_1-equivalence classes of loops based at x of length $\leq 2\max\{l-d,0\}+(4k-2)d+4k\delta$. Our assumption is that it does not exceed $k-1$. Therefore $V\leq\max\{2\max\{l-d,0\}+(4k-2)d, s\}+\epsilon+(2k-1)(\max\{l-d,0\}+(2k-1)d+2k\delta)$, where, first, δ and, then, ϵ can be made arbitrarily small. On the other hand $V\geq s_1+U>\max\{s_1+d, 2d+U\}$, and, thus, satisfies the conditions of Lemma 2.7. Finally, if ϵ and δ are sufficiently small, then V is still strictly less than the level of γ, and γ and $\{x\}$ correspond to different points of $P(U,V)$.

2.3.3. In order to complete the proof of Proposition 2.6 it remains to verify that $\psi : P(U,V) \longrightarrow M^n$ is a covering map. It follows directly from Definition 1.1 that ψ is the covering away from the boundary. But Lemma 2.7 implies the possibility to represent every element of $P(U,V)$ by a path of length $\leq U-\delta$. So, we are never near the boundary. More formally, in the considered case $P(U,V)=P(U-\delta,V)$. Also we can choose $\epsilon=\frac{1}{2}\min\{\epsilon_0,\delta\}$ in Definition 1.1. Now Definition 1.1 implies that ψ is a covering map in the usual sense. This completes the proof of Proposition 2.6. □.

2.4. Proof of Theorem 2.2. The proof of Theorem 2.2 follows the same ideas as the proof of Theorem 2.1. Yet we are able to choose smaller values of, first, U and, then, V in the specific situation of Theorem 2.2.

First, note that Lemma 2.5 implies the existence of a desired γ_1. We need to prove the existence of a geodesic loop γ_2 of length $\leq 6d$ that cannot be contracted to a point by a path homotopy passing via loops of length $\leq 6d$ such that γ_2 is $6d$-distinct from γ_1.

In fact, we are going to demonstrate that for all sufficiently small positive values of ϵ and $\delta<\epsilon/2$ if γ_1 cannot be contracted via loops of length $\leq 12d+2\delta+\epsilon$, then there is a geodesic loop γ_2 of length $\leq 6d+2\delta$ $(6d+\epsilon)$-distinct from γ_1 that cannot be contracted to a point via loops of length $\leq 6d+\epsilon$. Passing to the limit as $\delta\longrightarrow 0$ and using the Ascoli-Arzela theorem we will obtain the desired geodesic loop γ_2.

Let us proceed by contradiction. Assume that there is no such loop γ_2. Our first observation is that every path of length $\leq 3d+\delta$ starting at x can be connected by a path homotopy passing through paths of length $\leq 7d$ with a path of length $\leq 3d$. (In fact, this assertion will be true for all paths of length $\leq 5d$ starting at x, that is for all $\delta\leq 2d$.) This observation will play the same role in the proof of Theorem 2.2 as Lemma 2.7 played in the proof of Theorem 2.1.

Indeed, connect the endpoints of this path τ by a minimal geodesic m. If the resulting loop $\tau * m^{-1}$ of length $\leq 6d$ based at x can be contracted to x by a path homotopy passing through loops of length $\leq 6d$, then the desired path homotopy can be obtained by inserting $m^{-1}*m$, and contracting $\tau*m^{-1}$ to a point by the path homotopy that passes through loops of length $\leq 6d$. If $\tau*m^{-1}$ is not contractible to a point by a path homotopy that passes through loops of length $\leq 6d$, then it gets stuck at γ_1 or γ_1^{-1}. In this case τ can be connected with γ_1*m or $\gamma^{-1}*m$ by an obvious path homotopy passing through paths of length $\leq 7d$.

Now for every small positive ϵ and $\delta<\epsilon/2$ we choose $U=3d+\delta$ and $V=9d+\delta+\epsilon$. We claim that this value of V satisfies condition (*). Indeed, in order to verify (*) we need to check that every geodesic loop γ_0 based at x of length $\leq 6d+2\delta$ that is $(12d+2\delta+\epsilon)$-equivalent to the trivial loop is also $(6d+\epsilon)$-equivalent to the trivial loop. But our assumption implies that there is either one or two non-trivial $(6d+\epsilon)$-equivalence classes of loops based at x of length $\leq 6d+2\delta$, namely the $(6d+\epsilon)$-equivalence classes of γ_1 and γ_1^{-1} (which, in principle, can coincide). And all geodesic loops in this class (or these two classes) are not $(12d+2\delta+\epsilon)$-equivalent to the trivial loop.

Thus, $\psi : P(U,V)\longrightarrow M^n$ will be a covering away from the boundary. Since all elements of $P(U,V)$ are representable by paths of length $\leq 3d=U-\delta$, ψ will be a covering map in the usual sense. As M^n is simply connected, ψ must be a homeomorphism.

Yet ψ cannnot be a homeomorphism. Indeed, it maps the V-equivalence classes of γ_1 and the trivial loop into x. But γ_1 is not V-equivalent to the trivial loop, so these two points of $P(U, V)$ mapped to x are distinct. Thus, we obtain a contradiction. □

2.5. Simply-connected Riemannian manifolds with very deep local minima of the length functional on the loop spaces. The assumptions in Theorem 2.1 immediately lead one to the following question: Given a smooth simply-connected manifold M^n and a constant C are there Riemannian metrics on M^n such that for some $p \in M^n$ the depth of a non-trivial local minimum of the length functional on $\Omega_p M^n$ is not less than Cd, where d denotes the diameter of the Riemannian metric? The answer for this question is always positive. In this section we discuss two different constructions of such Riemannian metrics.

2.5.1. If $n = 2$, and M^n is diffeomorphic to S^2, we can construct the desired Riemannian metrics by gluing two copies of the Riemannian 2-discs constructed in the paper of S. Frankel and M. Katz ([FK], Proposition $2\frac{1}{2}$) along their boundaries.

The geometry of these metrics on the 2-disc can be described as follows. Start with a flat disc of radius one (or, alternatively, a disc with a large constant negative sectional curvature to simplify some technicalities in the proofs). Embed a binary tree of a very large height H into this disc, so that each edge of this tree has length ≥ 1. (The larger is H, the larger with be the depth of the resulting local minimum of the length functional.) Consider a simple closed curve that goes around this tree very close to it. Change a Riemannian metric in a tiny neighbourhood of this curve (disjoint from the tree) so as to build a "wall" of height one above this curve that will "separate" the tree from the rest of the disc. Now one can prove that there will be extremely long curves during each homotopy $\gamma_t,\ t \in [0, 1]$, that contracts the boundary of the disc γ_0 by a point observing that some of the closed curves γ_t must intersect the tree at a large number of points, and every time a curve intersects the tree it must first "climb" up and down the wall (which adds 2 to its length).

After we double these discs, take a point p on the common boundary γ of the two copies of the disc, and shorten γ to a local minimum of the length functional on the space of loops based at p by a length non-increasing homotopy, we obtain a very deep local minimum of the length functional.

It seems obvious that the idea of [FK] can be generalized to higher dimensions and can be used to produce such Riemannian metrics on each smooth simply-connected manifold of an arbitrary dimension.

2.5.2 If $n \geq 4$, then such examples can be obtained using a completely different idea: First, one constructs "short but tricky" finite presentations of the trivial group. ("Tricky" means here that it is extremely difficult to see that the group is, in fact, trivial. In particular, one needs an enormous amount of Tietze moves to transform the considered finite presentation to the trivial finite presentation of the trivial group. The existence of such finite presentations follows from the non-existence of an algorithm deciding whether or not a given finitely presented group is trivial.) Then one can use a version of the classical Dehn construction to realize the considered finite presentation as an "obvious" finite presentation of the fundamental group of a homology M^n constructed as a submanifold of an Euclidean space. By virtue of the properties of this construction the resulting manifold will be diffeomorphic to M^n. But it "looks" like a manifold with a non-trivial infinite fundamental group. Its Riemannian metric induced by the embedding in the Euclidean space will have the desired property: One can realize the generators of the fundamental group by "short" geodesic loops but one needs to increase their length enormously before they will become contractible to a point.

Yet note that it is difficult (albeit possible) to implement this idea in the just described direct way because of the technical difficulties explained in section 5.F of [N].

Instead the proof of Theorem 1 in [N] provides an indirect construction of such Riemannian metrics on every simply-connected manifold of dimension $n \geq 5$. In the notations of Theorem 1 the Riemannian metrics of interest to us correspond to points in "deep" non-trivial connected components of $Riem_\epsilon M^n$ for sufficiently small ϵ. The corresponding (simply-connected) Riemannian manifolds "look" like they have a non-trivial fundamental group. As no algorithm can tell us otherwise, the effective universal coverings $P(U, V)$ will not coincide with the underlying Riemannian manifold up to a very large value of U which will grow faster than any computable function of $\lfloor \frac{1}{\epsilon} \rfloor$.

The idea of [N] works not only for all smooth simply-connected manifolds of dimension $n \geq 5$ but also for all 4-dimensional manifold representable as the direct sum of a simply-connected manifold N^4 and a sufficient number of copies of $S^2 \times S^2$ (see section 5.A of [N]. The number 46 is sufficient here.) Conjecturally the approach of [N] should work for all simply-connected four-dimensional manifolds.

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