# AN EFFECTIVE ORBIFOLD GROUPOID IS DETERMINED UP TO MORITA EQUIVALENCE BY ITS UNDERLYING DIFFEOLOGICAL ORBIFOLD

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Abstract. A diffeology on a set $X$ declares which functions from the open subsets of Euclidean spaces to $X$ are differentiable. Orbifolds are represented by Lie groupoids that are locally isomorphic to the groupoids associated with finite group actions. We show that such a Lie groupoid, if effective, is determined up to Morita equivalence by the natural diffeology of its quotient space.

Contents



## 1. Introduction

In [IKZ], it was shown that diffeological orbifolds contain the same information as Satake's V-manifolds, which are the original definition of what are now called orbifolds. In this paper, we set out to show directly that diffeological orbifolds also contain the same information as effective proper étale groupoids (called "effective orbifold groupoids" in this paper), which is the other object that is usually taken as the definition of orbifolds. One reason for doing this (as opposed to referring to the literature for the correspondence between orbifolds and proper étale groupoids) is that diffeological orbifolds (unlike V-manifolds) form a category, and groupoids form a 2-category, and we would like to give a more categorical correspondence between these two objects by studying how the morphisms relate.

With this goal in mind, we consider the Morita equivalence relation on orbifold groupoids and diffeomorphisms between diffeological orbifolds and show

1.1. **Theorem.** *Two effective orbifold groupoids are Morita equivalent if and only if their quotient spaces are diffeomorphic as diffeological spaces.*

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The paper is organized as follows:

- In §2, we recall the definitions of Lie groupoids and diffeological spaces and show that there is a quotient functor from the category of Lie groupoids (and Lie groupoid morphisms) to the category of diffeological spaces.
- In §3, we recall the definition of Morita equivalence and show the "only if" part of the above theorem for all Lie groupoids: if two Lie groupoids are Morita equivalent then their quotient spaces are diffeomorphic. This implies that the map from Morita equivalence classes of Lie groupoids to diffeological spaces is well-defined.
- In §4, we give a definition for orbifold groupoids, which turns out to be equivalent to proper étale groupoids, but it has the advantage of being more obviously of a local nature; we also show that the quotient spaces of orbifold groupoids are diffeological orbifolds.
- In §5, we show that the quotient functor from §2 is surjective on objects when restricted to orbifold groupoids.
- In §6, we further restrict the category of Lie groupoids under consideration by defining the notion of effective orbifold groupoids and show that the arrows of an effective orbifold groupoid can be recovered from the germ of its action on the objects of the groupoid.
- In §7, we show that every diffeomorphism between diffeological orbifolds comes from a Morita equivalence between orbifold groupoids. This proves half of what is needed to conclude an equivalence of categories between effective orbifold groupoids (with Morita equivalences for arrows) and diffeological orbifolds (and diffeomorphisms for arrows).

The following two key facts are the main motivation for believing such a result to be plausible, and are essentially all that is used in the proofs, even though they appear in various guises. We will elaborate on these facts in much greater detail in the following sections, but for now we just want to give the reader an idea of what is the idea behind the paper:

- if we happen to know that a diffeological orbifold $X$ is diffeomorphic to the (global) quotient of $\mathbb{R}^n$ by two finite groups, which act linearly on $\mathbb{R}^n$, then the groups are isomorphic (cf. Lemma 5.4). Moreover, any diffeomorphism between two such quotients lifts to an equivariant diffeomorphism between the $\mathbb{R}^n$'s (this is not true for arbitrary smooth maps).
- every proper étale groupoid is locally the action groupoid of a finite group acting on an open subset of $\mathbb{R}^n$.

## 2. Lie groupoids and their underlying diffeology

Diffeology is one of several ways of applying differential geometry to spaces that are not manifolds. A diffeology on a set $X$ specifies which maps from open subsets of $\mathbb{R}^n$ to $X$ are declared to be differentiable. This is analogous to defining a manifold structure by providing smooth parametrizations of the elements of an open covering; however, because the maps that define a diffeology do not have to be injective, diffeology is well-suited for studying quotients. Another, "higher", approach to quotient spaces is through Lie groupoids. Every Lie groupoid has an underlying diffeological space. In this section we review the definitions of these notions, and we describe a forgetful functor from the category of Lie groupoids and their homomorphisms to the category of diffeological spaces and differentiable maps.

2.1. **Definition.** Let $X$ be a set. A **parametrization** on $X$ is a map from an open subset of $\mathbb{R}^n$, for some natural number $n$, to $X$. A **diffeology** on $X$ is a set of parametrizations $p\colon U \to X$, called **plots**, that satisfies the following axioms:

- **Covering axiom:** Constant maps are plots.
- **Locality axiom:** Given a parametrization $p\colon U \to X$, if each $x \in U$ is contained in an open subset $V_x \subset U$ such that $p|_{V_x}\colon V_x \to X$ is a plot, then $p$ is a plot.
- **Smooth compatibility axiom:** If $p\colon U \to X$ is a plot, $V$ is an open subset of $\mathbb{R}^m$, and $F\colon V \to U$ is smooth, then the composition $V \xrightarrow{F} U \xrightarrow{p} X$ is a plot.

A **diffeological space** is a set equipped with a diffeology.

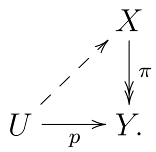
2.2. **Definition.** Let $X$ and $Y$ be diffeological spaces. A map $f\colon X \to Y$ is called **differentiable** if composition with $f$ takes plots of $X$ to plots of $Y$, i.e., for any plot $p\colon U \to X$, the composition $U \xrightarrow{p} X \xrightarrow{f} Y$ is a plot of Y. The map is a *diffeomorphism* if it is one-to-one and onto and if both $f$ and $f^{-1}$ are differentiable.

2.3. *Example.* A manifold is naturally a diffeological space. Its diffeology determines the manifold structure, i.e., it determines the equivalence class of atlases. A map between manifolds is differentiable in the diffeological sense if and only if it is smooth in the usual sense.

Diffeology is particularly suitable for handling quotient spaces.

2.4. **Definition.** Let $X$ be a diffeological space, $\sim$ an equivalence relation on $X$, and $\pi\colon X \to Y = X/\sim$ the quotient map. The **quotient diffeology** on $Y$ is defined by the maps that locally lift to plots of $X$. That is, a map $U \xrightarrow{p} Y$ is a plot if and only if for every point $p \in U$ there exist an open neighbourhood $V$ and a plot $V \xrightarrow{\tilde{p}} X$ such that $p|_V = \pi \circ \tilde{p}$. We express this using the following diagram:

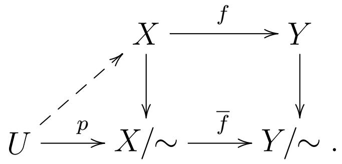
(2.1)




The dashed arrow represents the local liftings of $p$, which are defined on open subsets $V \subset U$ that cover $U$. Note that the sets $V$ themselves do not appear explicitly in the diagram. We will use similar diagrams throughout the paper.

2.5. **Lemma.** *Let $X$ and $Y$ be diffeological spaces, equipped with equivalence relations, and let $f\colon X \to Y$ be a differentiable map. Suppose that, for every $x_1$ and $x_2$ in $X$, if $x_1 \sim x_2$ then $f(x_1) \sim f(x_2)$. Then the induced map $\overline{f}$ from $X/\sim$ to $Y/\sim$ is differentiable with respect to the quotient diffeologies.*

*Proof.* By the definition of "differentiable map" and of the quotient diffeology, we need to show that if a map $p\colon U \to X/\sim$ admits local liftings to $X$ then the composition $\overline{f} \circ p\colon U \longrightarrow Y/\sim$ admits local liftings to $Y$. Such local liftings are obtained by composing $f$ with local liftings of $p$:



∎

We now recall the definition of a Lie groupoid. We refer the reader to [MM] for further elaboration on the subject.

Recall that a groupoid is a small category ("small" means that the space of objects forms a set) in which every arrow is invertible. A Lie groupoid is a groupoid with extra structure: the objects and arrows are manifolds, and the structure maps are smooth.

The definition of Lie groupoid, and the definition of pseudo-equivalence that appears in section 3, involve surjective submersions. The following characterization of submersions will be more useful for us than the straightforward definition of submersion, and will allow generalizations to maps with local properties other than being $C^\infty$. Haefliger terminology?

2.6. **Lemma.** *A smooth map between manifolds, $f\colon M \to N$, is a submersion if and only if it has local sections through each point: for every point in $M$ there exists an open neighborhood $V \subset N$ and a smooth map $s\colon V \to M$ such that $f \circ s = identity|_V$.*

*Proof.* This follows from the implicit function theorem. ∎

We express a surjective submersion $M \to N$ using the following diagram.

(2.2)

Note that, although the notation is similar to that of diagram (2.1), the condition here is stronger than that in the diagram (2.1): local sections exist through each point of $M$ and not just on neighborhoods of each point of $N$.

2.7. **Definition.** A **Lie groupoid** $G = \begin{array}{c} G_1 \\ {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} \\ G_0 \end{array}$ consists of a manifold $G_0$ whose points are the *objects*, and a manifold $G_1$ whose points are the *arrows*, and the following structure maps.

(i) A *source map* $s\colon G_1 \to G_0$, which is a surjective submersion, and a *target map* $t\colon G_1 \to G_0$. (See part (2) of Remark 2.8.)

Thus, $g \in G_1$ represents an arrow from $s(g)$ to $t(g)$. For any two objects $x, y \in G_0$, we denote by $G_{x,y}$ the set of arrows from $x$ to $y$:

$$G_{x,y} = \{g \in G_1 \,|\, s(g) = x,\ t(g) = y\}.$$

The *space of composable arrows* is the fibered product

$$G_1 \times_{G_0} G_1 = \{(h, g) \in G_1 \times G_1 \,|\, t(g) = s(h)\}.$$

It is a manifold, because $s$ is a surjective submersion. We have the following Cartesian diagram:

$$\begin{array}{ccc} G_1 \times_{G_0} G_1 & \longrightarrow & G_1 \\ \downarrow & & \downarrow t \\ G_1 & \xrightarrow{s} & G_0 \end{array}$$

(ii) A *multiplication map*

$$\begin{aligned} m\colon G_1 \times_{G_0} G_1 &\longrightarrow G_1 \\ (h, g) &\longmapsto hg, \end{aligned}$$

(iii) a unit map

$$\begin{aligned} u\colon G_0 &\longrightarrow G_1 \\ x &\longmapsto 1_x \end{aligned},$$

(iv) and an inverse map

$$\begin{aligned} i\colon G_1 &\longrightarrow G_1 \\ g &\longmapsto g^{-1} \end{aligned}.$$

These maps are required to be smooth and to satisfy the algebraic properties of a groupoid: Multiplication is associative:

$$(hg)f = h(gf);$$

the unit $1_x \in G_{x,x}$ is a neutral element for left or right multiplication:

$$1_y g = g 1_x = g \text{ for any } g \in G_{x,y};$$

for any $g \in G_{x,y}$, its inverse $g^{-1} \in G_{y,x}$ satisfies

$$gg^{-1} = 1_y \quad \text{and} \quad g^{-1}g = 1_x.$$

2.8. *Remark.* (1) In this paper, all manifolds are assumed to be Hausdorff and second countable. These conditions are relaxed in some other treatments of Lie groupoids, such as [BX].

(2) Because the inverse map is an involution that intertwines the source map with the target map, the source map being a surjective submersion implies that the target map is a surjective submersion too.

The simplest example of a groupoid is that of a manifold $M$, with all the arrows being given by the identity map. Another example, which is valuable to keep in mind when working with orbifolds, is a groupoid that is associated to an open covering of a manifold. Eventually we would like to consider these two groupoids equivalent to each other; this motivates the definitions of section 3.

check terminology: Čech groupoid, identity groupoid.

2.9. *Example.* Let $M$ be a manifold. Its identity groupoid look up how it's called is the groupoid in which $G_0 = G_1 = M$ and the source and target maps are the identity map.

2.10. *Example.* Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of a manifold $M$. terminology: covering groupoid? (Eugene) Its **Čech groupoid** is

$$\Gamma(\mathcal{U}) := \begin{array}{c} \coprod U_{ij} \\ {\scriptstyle s}\downdownarrows{\scriptstyle t} \\ \coprod U_i, \end{array}$$

where $U_{ij} = U_i \cap U_j$. The top union is over the set $I \times I$ of ordered pairs $(i, j)$ of indices in $I$; the order is needed to define the structure maps. (Set theoretically, to make the unions disjoint, we may take the sets $U_{ij} \times \{(i,j)\}$ and $U_i \times \{i\}$.) The source and target maps are induced from the natural inclusion maps $s \colon U_{ij} \hookrightarrow U_i$ and $t \colon U_{ij} \hookrightarrow U_j$. Note that these maps, as maps from $\bigsqcup U_{ij}$ to $\bigsqcup U_i$, are surjective submersions. The remaining structure maps are defined in the obvious way. The identity groupoid is the special case of a Čech groupoid when the open covering contains the single element, $M$.

We will later see that the Čech groupoid is étale; it is proper if the covering is locally finite.

Another important example is a groupoid that is associated to a group action on a manifold; this groupoid is defined in the following example and is illustrated in Figure 1.

2.11. *Example.* Let a Lie group $K$ act on a manifold $M$. The **action groupoid** is

$$K \ltimes M := \begin{array}{c} K \times M \\ {\scriptstyle s}\downdownarrows{\scriptstyle t} \\ M, \end{array}$$

where the source map $s$ is the projection onto the $M$ factor, and where the target map $t$ is the action map $(k, m) \mapsto k \cdot m$.

A simple example of such a groupoid, associated to the group $\mathbb{Z}/3\mathbb{Z}$ acting on the unit disc by rotations, is illustrated in Figure 1. Here, the source map is given by rotating the top sheets so the rays line up and then projecting down, while the target map is given by projecting the top sheets down as shown in the figure. The action groupoid should be thought of as the "movie" of the action.

Let $\begin{array}{c} G_0 \\ {\scriptstyle s}\downdownarrows{\scriptstyle t} \\ G_1 \end{array}$ be a Lie groupoid. Define a relation on objects $G_0$, by declaring that $x \sim y$ if and only if there exists an arrow in $G_1$ whose source is $x$ and whose target is $y$. The groupoid axioms guarantee that this is an equivalence relation. The quotient, which we denote $QG = G_0/G_1$, with its natural quotient diffeology, is the **underlying diffeological space** of the groupoid.

For the groupoid associated to an open covering of a manifold $M$, the quotient is naturally identified, as a diffeological space, with the manifold $M$. For the action groupoid associated to an action of a group $G$ on a manifold $M$, the quotient is naturally identified, as a diffeological space, with the quotient $M/G$. need to check these facts?

Homomorphism of Lie groupoids is defined in a natural way:

FIGURE 1. An action groupoid corresponding to a finite group action.

2.12. **Definition.** Let $G = \begin{array}{c} G_1 \\ {\scriptstyle s}\downdownarrows{\scriptstyle t} \\ G_0 \end{array}$ and $H = \begin{array}{c} H_1 \\ {\scriptstyle s'}\downdownarrows{\scriptstyle t'} \\ H_0 \end{array}$ be Lie groupoids. A **homomorphism** $\phi\colon G \to H$ consists of two smooth maps $\phi_1\colon G_1 \to H_1$ and $\phi_0\colon G_0 \to H_0$, (which are sometimes denoted by the same symbol $\phi$,) that form a functor from the underlying category of $G$ to the underlying category of $H$. A homomorphism $\phi$ is an *isomorphism* if the maps $\phi_0$ and $\phi_1$ are one-to-one and onto and their inverses are smooth.

A homomorphism can be denoted by the following diagram:

$$\begin{array}{ccc} G_1 & \xrightarrow{\phi_1} & H_1 \\ {\scriptstyle s}\downdownarrows{\scriptstyle t} & & {\scriptstyle s'}\downdownarrows{\scriptstyle t'} \\ G_0 & \xrightarrow[\phi_0]{} & H_0. \end{array}$$

2.13. **Proposition.** *A homomorphism of Lie groupoids $G \to H$ naturally descends to a differentiable map between their quotient diffeological spaces, $G_0/G_1 \to H_0/H_1$. This gives a "forgetful functor"*

$$Q : G = \begin{array}{c} G_1 \\ {\scriptstyle s}\downdownarrows{\scriptstyle t} \\ G_0 \end{array} \mapsto QG = G_0/G_1$$

*from the category of Lie groupoids and homomorphisms to the category of diffeological spaces and differentiable maps.*

*Proof.* This follows from Lemma 2.5. do we need more detail? □

might want to clarify the relation with the "differentiable space" notion of [BX]. See their section 2.7. Their notion is different: e.g. they allow quotients of free non-proper actions but they don't allow quotients of free actions. In particular should explain diffeology as a "sheaf" as Eugene said.

## 3. Pseudo-equivalences of Lie groupoids descend to diffeomorphisms of diffeological spaces

Isomorphism of groupoids, as in Definition 2.12, turns out to be too restrictive. For example, we would like to represent a manifold by the Čech groupoid that is associated to an open covering, and to consider groupoids that are associated to different coverings of the same manifold as equivalent to each other. However, if $G$ and $H$ are the groupoids that correspond to different coverings of the same manifold, generally there do not exist morphisms of groupoids $G \to H$ and $H \to G$ that are inverses to each other. The simplest example is if $G$ is the covering groupoid of a connected manifold $M$ with a single open set (i.e., $M$ itself) and $H$ is the covering groupoid of $M$ associated to a covering by two open sets, $\mathcal{U} = \{U, V\}$, that are non-empty and not equal to $M$. These two groupoids are supposed to represent the same geometric object. We would like the natural morphism $H \to G$, which sends $U \sqcup V$ onto $M$ using the inclusion maps of $U$ and of $V$ into $M$, to induce an equivalence relation. However, as a morphism of Lie groupoids, this morphism is not invertible, because the natural map $U \sqcup V \to M$ does not have a smooth inverse. The cure for this is to define a coarser notion of equivalence, obtained from maps of groupoids such as those that represent refinements of covers.

maybe, instead, talk about *refinement* of covers.

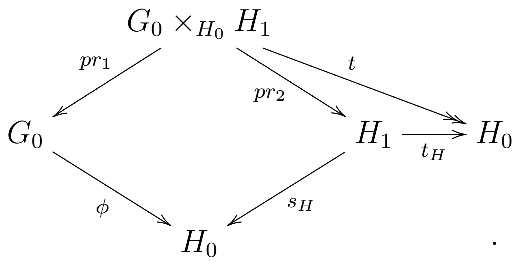
Algebraically, the relevant property of the map $U \bigsqcup V \to M$ is that it gives an *equivalence of categories*: it is essentially surjective (in this case it is surjective) and fully faithful. Essential surjectivity of the map $H \to G$ means that every object in $G$ is isomorphic to some object in the image of $H_0 \to G_0$. Fully faithfulness means that for any two objects $x, y \in H_0$, with images $x', y' \in G$, the map $H_{x,y} \to G_{x',y'}$ is a bijection.

*Pseudo-equivalence* is an adaptation of equivalence of categories to the context of Lie groupoids. The relevant maps are required to be smooth and surjective maps to be submersions as well. The meaning of the surjective submersion requirement is, as explained earlier, that the map admits local sections through every point.

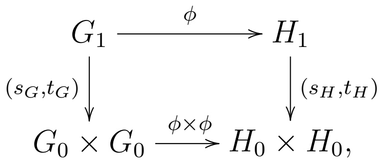
The reason for the terminology "pseudo-equivalence" is that the existence of such a map is not an equivalence relation on groupoids. See Example 3.2. The equivalence relation that is induced by pseudo-equivalences, which is called *Morita equivalence*, turns out to be the relevant equivalence relation between Lie groupoids. See Definition 3.4.

3.1. **Definition.** A morphism $\phi \colon G \to H$ from a Lie groupoid $G = \begin{matrix} G_1 \\ {\scriptstyle s_G}\downarrow\downarrow{\scriptstyle t_G} \\ G_0 \end{matrix}$ to a Lie groupoid $H = \begin{matrix} H_1 \\ {\scriptstyle s_H}\downarrow\downarrow{\scriptstyle t_H} \\ H_0 \end{matrix}$ is a **pseudo-equivalence** if the following two conditions are satisfied.

(ES) The map $t' \circ pr_2$ in the following commutative diagram is a surjective submersion:




(FF) The following square is a fibred product of manifolds,




that is, the induced map

$$G_1 \to (G_0 \times G_0) \times_{(H_0 \times H_0)} H_1$$

is a diffeomorphism.

3.2. *Examples.* .

(1) Let $G$ be the Čech groupoid associated to an open covering $\mathcal{U}$ of a manifold $M$. Let $H$ be the identity groupoid associated to the manifold $M$. Then the natural morphism $G \to H$ is a pseudo-equivalence. However, except in trivial cases, there does not exist a pseudo-equivalence from $H$ to $G$, for the same reason as in the case of groupoid isomorphisms: that would require a map $H \to G$, which sometimes doesn't exist as explained above.

(2) Let $P \to M$ be a principal $K$-bundle for a Lie group $K$. Let $I_M$ denote the identity groupoid associated to the base, $M$. Then the natural morphism $K \ltimes P \to I_M$ is a pesudo-equivalence.

3.3. **Proposition.** *If $\phi\colon G \to G'$ is a pseudo-equivalence of groupoids, then the map induced on the quotients, $Q\phi\colon G_0/G_1 \to H_0/H_1$, is a diffeomorphism of diffeological spaces.*

*Proof.* We need to show that $Q\phi$ is surjective and injective and that the inverse map, $(Q\phi)^{-1}$, is differentiable. This is done by chasing through the diagrams that appear in the definition of pseudo-equivalence.

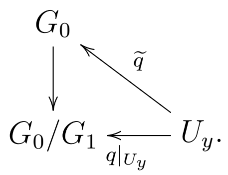
Consider the diagram (ES). Because the map $t' \circ pr_2$ is surjective, for any $y \in H_0$ there exists $(x, g') \in G_0 \times H_1$ that maps to it; this means that $g'$ is an arrow from $\phi_0(x)$ to $y$. Thus, $[y]$ is equal to $[\psi_0(x)$ in the quotient $H_0/H_1$. By the definition of the map on the quotient spaces, $[y] = Q\phi([x])$, so $[y] \in \text{image}Q\phi$. Since $y \in H_0$ was arbitrary, this shows that $Q\phi$ is onto.

Consider the diagram (FF). Take any $x_0, x_1 \in G_0$; let $y_0 = \phi(x_0)$ and $y_1 = \phi(x_1)$. The map $Q\phi$ takes the elements $[x_0]$ and $[x_1]$ of $G_0/G_1$ to the elements $[y_0]$ and $[y_1]$ of $H_0/H_1$. We need to show that if $[y_0] = [y_1]$ then $[x_0] = [x_1]$. The equation $[y_0] = [y_1]$ means that the set of arrows $H_{y_0,y_1}$ from $y_0$ to $y_1$ is non-empty. The condition in (FF) implies that the map $\phi\colon G_{x_0,x_1} \to H_{y_0,y_1}$ is a bijection. In particular, if $H_{y_0,y_1}$ is nonempty, then $G_{x_0,x_1}$ is nonempty, which means that $[x_0] = [x_1]$.

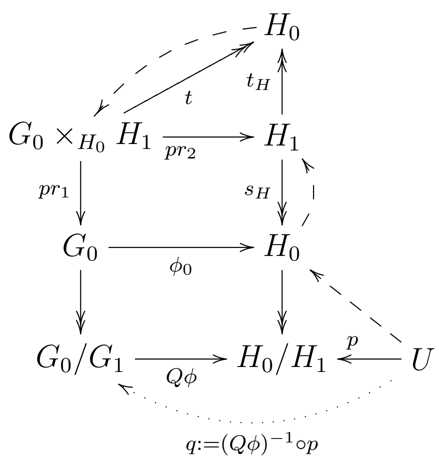
Since $Q\phi$ is a bijection of sets, it has an inverse map, $(Q\phi)^{-1} : H_0/H_1 \to G_0/G_1$. It remains to show that this map is differentiable, ie for each plot $p : U \to H_0/H_1$, the composition

$$q := (Q\phi)^{-1} \circ p : U \to H_0/H_1 \to G_0/G_1$$

is also a plot. By locality of diffeologies, it is enough to show that for each $y \in U$, there is a neighbourhood $U_y$ of $y$ such that $q\big|_{U_y}$ is a plot for $G_0/G_1$. Now, by the definition of the quotient diffeology, this means that $\exists\ \widetilde{q} : U_y \to G_0$ such that the following diagram commutes:

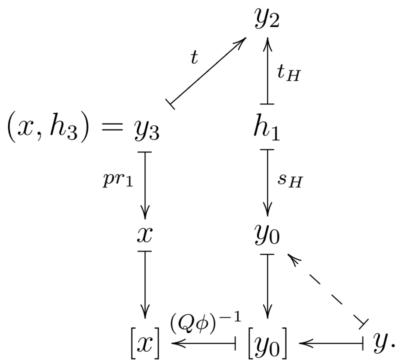



What we will use is the fact that $p : U \to H_0/H_1$ is a plot and so (by the definition of the quotient diffeology on $H_0/H_1$) it locally lifts to $H_0$, as well as the (ES) condition of pseudo-equivalence, both of which appear in the following diagram (Note: the top half is the (ES) diagram rotated by 45°):




As explained earlier, the dashed arrows designate local lifts, defined on small open subsets.

Now, given $y \in U$ choose a local lift $\widetilde{p} : U_y \to H_0$ where $U_y$ is a neighbourhood of $y$, and call $y_0 := \widetilde{p}(y)$; then choose $h_1 \in s_H^{-1}(y_0)$, denote $y_2 := t_H(h_1)$, and again choose $y_3 := (x, h_3) \in t^{-1}(y_2)$: by definition, $x := pr_1(y_3)$. These relations are depicted in the following diagram:




The main thing to note here is that $[x] = (Q\phi)^{-1}[y_0]$ because of the following: we have the maps $y_0 \xleftarrow{s_H} h_1 \xrightarrow{t_H} y_2$ and the equality $y_2 = t(y_3) := t_H \circ pr_2(x, h_3) := t_H(h_3)$, so we get $[y_0] = [y_2] \in H_0/H_1$ (by the definition of $H_0/H_1$) and $[y_2] = [t_H(h_3)]$; also, as with $h_1$, we have $[t_H(h_3)] = [s_H(h_3)] \in H_0/H_1$, which together with $\phi(x) = s_H(h_3)$ (by the fact that $y_3 = (x, h_3) \in G_0 \times_{H_0} H_1$) gives us the equality $Q\phi[x] = [y_0]$, or in other words

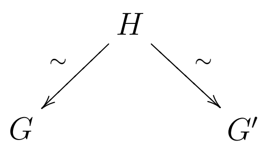
$$(Q\phi)^{-1}[y_0] = [x].$$

Also, note that for this last conclusion, we only used the existence of the appropriate preimages for $y_0$ as drawn in the previous diagram: this is important because next we will extend the above diagram to the points in a small neighbourhood of $y \in U$ and we want to make sure the same conclusion holds for these other points.

This is done as follows: shrink $U_y$ as needed to make sure $s_H$ has a section, $\sigma_1$, through $h_1$ over $\widetilde{p}(U_y)$ and $t$ has a section $\sigma_2$ through $y_3$ over the image of $U_y$ under $t \circ \sigma \circ \widetilde{p}$ (these sections, together with $\widetilde{p}$, are what is denoted by dashed arrows in the first diagram in this proof). Using these sections we get a map $\widetilde{q} : U_y \to G_0$ which we claim to be a lift of $q\big|_{U_y} := (Q\phi)^{-1} \circ p\big|_{U_y}$: here we use the above observation that was made after concluding that $[x] = (Q\phi)^{-1}[y_0]$.

Since $y \in U$ was arbitrary, this shows that $q$ is a plot for $G_0/G_1$, since it locally lifts to $G_0$. □

3.4. **Definition.** A **Morita equivalence** between Lie groupoids $G$ and $G'$ is a third Lie groupoid $H$ and pseudo-equivalences

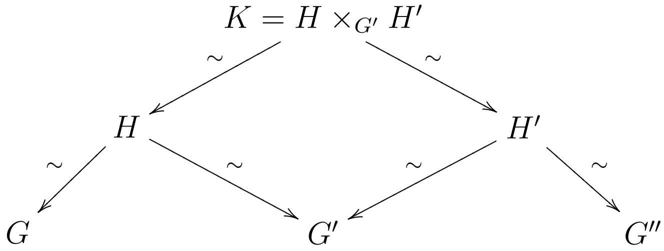



Two groupoids $G$ and $G'$ are **Morita equivalent** if there exists a Morita equivalence between them.

By Proposition 3.3, a Morita equivalence of groupoids induces a diffeomorphism of the quotients as diffeological spaces.

3.5. **Lemma.** *Morita equivalence is an equivalence relation.*

*Proof.* The only non-trivial thing to show is that Morita equivalence is transitive. Let $G \leftarrow H \to G'$ and $G' \leftarrow H' \to G''$ be Morita equivalences. By Prop. 5.12(iv) of [MM], the "weak fibred product" of $H$ and $H'$ over $G'$ (see [MM, §5.3]) provides a Morita equivalence between $G$ and $G''$:




□

## 4. Orbifold groupoids and their underlying diffeological orbifolds

need to modify first paragraph because of our current def of "orbifold groupoid":

An orbifold is a space modeled on finite quotients of $\mathbb{R}^n$. There are different ways of making sense of an orbifold as a global object. Modern treatments represent orbifolds by *proper étale Lie groupoids*. *Diffeological orbifolds* were introduced in [IKZ] and were shown there to be equivalent to Satake's notion of a V-manifold, which is the original definition of "orbifold" in the literature. In this section we define these concepts, and we show that the functor that associates to a Lie groupoid its underlying diffeological space restricts to a functor from orbifold groupoids to diffeological orbifolds.

It's not clear if at this point the arrows on groupoids should be homomorphisms or weak maps; we will decide on this once we clarify what it is that we are proving. In this section we do not yet need "effective".

We will define diffeological orbifolds to be diffeological spaces that locally look like quotients of $\mathbb{R}^n$ by a finite group. First we must make sense of "locally". In the definition of diffeology, we begin with a set, not a topological space. However, the diffeology naturally induces a topology; we will use this topology whenever we refer to open subsets of a diffeological space.

4.1. **Definition.** The **D-topology** on a diffeological space $X$ is

$$\{U \subset X \,|\, p^{-1}(U) \text{is open for any plot } p\colon V \to X\}.$$

Note that the D-topology is the biggest (coarsest) topology (in the sense that it has the fewest number of open sets) for which all the plots are continuous. Also, note that any differentiable map of diffeological spaces is continuous. (Thus, there is a natural forgetful functor that takes a diffeological space to its underlying topological space.)

4.2. **Definition.** A diffeological space $X$ is called a **diffeological orbifold** if each point $x \in X$ has a neighbourhood $U \ni x$ which is diffeomorphic (as diffeological spaces) to an open subset of $\mathbb{R}^n/\Gamma$ for some $n \in \mathbb{N}$ and for some finite subgroup $\Gamma$ of $\mathrm{GL}_n(\mathbb{R})$.

From the groupoids point of view, an orbifold is represented by a Lie groupoid that is locally isomorphic to the action groupoid of a finite group action. To define this, we first need to define what we mean by "locally".

Let $G = \begin{smallmatrix} G_1 \\ s\downarrow\downarrow t \\ G_0 \end{smallmatrix}$ be a Lie groupoid, and let $U \subset G_0$ be an open subset. Define $G|_U$ to be the groupoid whose set of objects is $U$ and whose set of arrows is $(s \times t)^{-1}(U \times U)$, that is,

$$(G|_U)_0 = U \quad , \quad (G|_U)_1 = \{g \in g_1 \mid s(g) \in U \text{ and } t(g) \in U\}.$$

This set of arrows is a manifold because it is an open subset of the manifold $G_1$. The structure maps of $G|_U$ are induced from those of $G$; the Lie groupoid axioms for $G|_U$ follow from those for $G$.

4.3. **Definition.** A Lie groupoid $G = \begin{smallmatrix} G_1 \\ s\downarrow\downarrow t \\ G_0 \end{smallmatrix}$ is an **orbifold groupoid** if for each point $p$ of $G_0$ there exists an open neighborhood $U \subset G_0$, and a finite group $K$ acting on $U$, and a Lie groupoid isomorphism

$$G|_U \cong K \ltimes U.$$

Need to confirm that Moerdijk defines an orbifold groupoid to be a proper étale Lie groupoid.

4.4. **Proposition** ([MM], Prop. 5.30)**.** *Let $G$ be a proper étale groupoid. Then any $x \in G_0$ has an open neighbourhood $U$ in $G_0$ with an action of the isotropy group $G_x$ such that there is an isomorphism of étale groupoids*

$$G|_U \simeq G_x \ltimes U.$$

Claim: every orbifold groupoid is étale.

Question: is every orbifold groupoid proper?

(Note: a group action which is locally proper might not be proper. In particular, being a proper group action is not a local condition. But for a finite group the action is always proper.)

Moerdijk, Lemma 5.20: discrete isotropy groups iff weakly equivalent to étale. Moerdijk, Lemma 5.26: if $G$ is proper and $G$ and $H$ are weakly equivalent then $H$ is proper.

If the stabilizers are finite then is the groupoid Morita equivalent to a proper étale Lie groupoid?

4.5. **Lemma.** *Let $\begin{smallmatrix} G_1 \\ s\downarrow\downarrow t \\ G_0 \end{smallmatrix}$ be a Lie groupoid. Then the quotient map $G_0 \xrightarrow{\pi} G_0/G_1$ is open.*

*Proof.* Let $U \subset G_0$ be an open subset. We need to show that $\pi(U)$ is an open subset of $G_0/G_1$.

By definition of the equivalence relation, we have $\pi^{-1}(\pi(U)) = t(s^{-1}(U))$. Because $s\colon G_1 \to G_0$ is continuous, $s^{-1}(U)$ is an open subset of $G_1$. Because $t\colon G_1 \to G_0$ is a submersion, it is an open map, so $t(s^{-1}(U))$ is an open subset of $G_0$. Thus, $\pi^{-1}(\pi(U))$ is open in $G_0$. By definition of the quotient topology, this means that $\pi(U)$ is open in $G_0/G_1$. □

4.6. **Lemma.** *If $G = \begin{smallmatrix} G_1 \\ s\downarrow\downarrow t \\ G_0 \end{smallmatrix}$ is an orbifold groupoid then $QG = G_0/G_1$ is a diffeological orbifold.*

*Proof.* Let $[x]$ be any point of $G_0/G_1$, and let $x \in G_0$ be a representative of this point. By definition of orbifold groupoid, there exists a neighborhood $U$ of $x$ in $G_0$ and a finite group $\Gamma$ acting on $U$ and an isomorphism of $G|_U$ with the action groupoid $\Gamma \ltimes U$. Let $QU$ be the image of $U$ in $G_0/G_1$. By Lemma 4.5, $QU$ is an open subset of $G_0/G_1$. By the definition of $G|_U$, the restriction to $U$ of the quotient map $G_0 \to G_0/G_1$ descends to a map

$$U/\Gamma \to QU \subset G_0/G_1 \tag{4.1}$$

which is one-to-one and onto. It remains to show that this map is differentiable and that its inverse is differentiable.

The fact that the map (4.1) is differentiable follows from the following diagram and the definition of quotient diffeology.

$$\begin{array}{ccccc} U & \xrightarrow{\text{identity}} & U & \xrightarrow{\text{inclusion}} & G_0 \\ \downarrow & & \downarrow & & \downarrow \\ U/\Gamma & \longrightarrow & QU & \longrightarrow & G_0/G_1 \end{array}$$

The fact that the inverse of the map (4.1) is differentiable follows from the same diagram and the fact that the map $U \to G_0$ is an open embedding. ☐

Thus, we have a natural forgetful functor from the category of orbifold groupoids and weak maps to the category of diffeological orbifolds and diffeomorphisms.

In section 5 we will show that this functor is essentially surjective.

section 7 should eventually show that this functor is fully faithful.

## 5. Every diffeological orbifold comes from an orbifold groupoid

We aim to clarify the relationship between the diffeological approach to orbifolds and the approach through groupoids. We described a functor, which we denote $Q$, from orbifold groupoids to diffeological orbifolds. In this section we show that this functor is essentially surjective. For this we need to show that every diffeological orbifold is diffeomorphic to the quotient space of some orbifold groupoid. To do this we take what we call an *orbifold atlas* $\mathcal{U} = \{(U_i, G_i, \phi_i)\}_{i \in I}$ and we look at its **germ groupoid**, $\Gamma(\mathcal{U})$. We define these below.

The main facts about diffeological orbifolds, which allows one to show that they contain the same information as orbifolds in the usual sense are collected in the following (perhaps somewhat surprising) lemmas which, in particular, allow one to recover the stabilizer groups of the orbifold from the diffeological structure alone. We will make use of these lemmas repeatedly in the following sections, as they are the heart of the matter; we refer the reader to [IKZ] for the proofs.

5.1. **Lemma.** *If $X$ is a diffeological orbifold, for every point $x$ of $X$ there exists a finite subgroup $\Gamma$ of $\mathrm{GL}_n(\mathbb{R})$ and a map $\varphi\colon \mathbb{R}^n \to X$ that takes $0$ to $x$ and that induces a diffeomorphism of $\mathbb{R}^n/\Gamma$ with a neighborhood of $x$ in $X$.*

*Proof.* See [IKZ, Lemma 14]. ☐

5.2. **Lemma.** *If $\Gamma \leq \mathrm{GL}_n(\mathbb{R})$ is a finite group, and $U \subseteq \mathbb{R}^n$ a connected open subset, and $h\colon U \to \mathbb{R}^n$ a $C^1$-map that preserves $\Gamma$-orbits (i.e., $h(x) \in \Gamma \cdot x$), then $\exists!\ \gamma \in \Gamma$ such that*

$$h(x) = \gamma \cdot x \quad \forall\, x \in U.$$

*Proof.* See [IKZ, Lemma 17]. ☐

This lemma is then used in [IKZ] to prove the following important fact about the existence of liftings of maps, on which rely most of our results.

5.3. **Lemma.** *Let* $\Gamma \leq \mathrm{GL}_n(\mathbb{R})$ *and* $\Gamma' \leq \mathrm{GL}_{n'}(\mathbb{R})$ *be finite subgroups. Let* $U' \subseteq \mathbb{R}^{n'}$ *be a* $\Gamma'$*-invariant open subset, and let*

$$f \colon \mathbb{R}^n/\Gamma \to U'/\Gamma'$$

*be a diffeomorphism. Then* $f$ *has a lift*

$$\tilde{f} \colon \mathbb{R}^n \to U'.$$

*Moreover, there is an isomorphism* $\phi\colon \Gamma \to \Gamma'$*, such that* $\tilde{f}$ *is a* $\phi$*-equivariant diffeomorphism of* $\mathbb{R}^n$ *with a connected component of* $U'$*.*

*Proof.* See [IKZ, Lemma 23]. □

5.4. **Lemma.** *Given two finite subgroups* $\Gamma \leq \mathrm{GL}_n(\mathbb{R})$ *and* $\Gamma' \leq \mathrm{GL}_{n'}(\mathbb{R})$*, and open subsets* $V \subseteq \mathbb{R}/\Gamma$ *and* $V' \subseteq \mathbb{R}^{n'}/\Gamma'$ *that contain the (image of the) origin, if there is a diffeomorphism* $\phi\colon V \to V'$ *with* $\phi(0) = 0$*, then we have to have* $n = n'$ *and* $\Gamma$ *and* $\Gamma'$ *are in the same conjugacy class in* $\mathrm{GL}_n(\mathbb{R})$*; in particular,* $\Gamma \cong \Gamma'$*.*

*Proof.* See [IKZ, Lemma 24]. □

Let $X$ be a diffeological orbifold.

By the definition of diffeological orbifold, for each point of $X$ there exists a neighborhood that is diffeomorphic to an open subset of the quotient of $\mathbb{R}^n$ by a finite subgroup of $\mathrm{GL}_n(\mathbb{R})$.

Fix a collection $\mathcal{U} = \{(U_i, G_i, \phi_i)\}_{i \in I}$ where each $G_i$ is a finite subgroup of $\mathrm{GL}_n(\mathbb{R})$, where $U_i$ is a $G_i$-invariant open subset of $\mathbb{R}^n$, where $\phi_i\colon U_i \to X$ induces a diffeomorphism from $U_i/G_i$ to an open subset of $X$, and where the images of the $\phi_i$s cover $X$.

We will now define the groupoid of germs of local diffeomorphisms corresponding to the atlas $\mathcal{U} = \{(U_i, G_i, \phi_i)\}_{i \in I}$. This groupoid is denoted $\Gamma(\mathcal{U})$. Credit goes to Haefliger. The set of objects is $\bigsqcup_{i \in I} U_i$. An arrow from an element $x$ of $U_i$ to an element $y$ of $U_j$ is the germ at $x$ of a diffeomorphism $f$ from a neighborhood $U'$ of $x$ in $U_i$ to a neighborhood $V'$ of $y$ in $U_j$ such that $\phi_i = \phi_j \circ f$. also refer to this construction in Moerdijk's book.

We would like to give $\Gamma(\mathcal{U})$ the structure of a Lie groupoid. The algebraic operations are defined in an obvious manner, and $\Gamma(\mathcal{U})_0$ is naturally a manifold, but we need to give $\Gamma(\mathcal{U})_1$ the structure of a manifold. With this structure, we need to prove that $\Gamma(\mathcal{U})$ is an orbifold groupoid and to show that the natural map from $\Gamma(\mathcal{U})$ to $X$ induces a diffeomorphism from the quotient space of $\Gamma(\mathcal{U})$ to $X$.

Fix any $i, j \in I$, open subset $U' \subset U_i$, open subset $V' \subset U_j$, and diffeomorphism $f\colon U' \to V'$ such that $\phi_j \circ f = \phi_i$. Define a map

$$\psi\colon U' \to \Gamma(\mathcal{U})_1$$

by

$$x \mapsto \mathrm{germ}_x f.$$

Note that this map is one-to-one. We will show the following:

5.5. **Proposition.** *There exists a unique manifold structure on* $\Gamma(\mathcal{U})_1$ *such that the maps* $\psi\colon U' \to \Gamma(\mathcal{U})_1$ *are open embeddings.*

look up criterion in Brocker+Janich

*Proof.* By the definition of $\Gamma(\mathcal{U})_1$, the images of the maps $\psi\colon U' \to \Gamma(\mathcal{U})_1$ cover $\Gamma(\mathcal{U})_1$. It is enough to show that for any two such maps, $\psi\colon U' \to \Gamma(\mathcal{U})_1$ and $\tilde{\psi}\colon \tilde{U}' \to \Gamma(\mathcal{U})_1$, the set $W = \psi^{-1}\tilde{\psi}(\tilde{U}')$ is open in $U'$, and the map $\tilde{\psi}^{-1} \circ \psi\colon W \to \tilde{U}'$ is smooth.

Consider another such map, $\tilde{\psi}$, obtained from $\tilde{i}, \tilde{j} \in I$, from open subsets $\tilde{U}' \subset U_{\tilde{i}}$ and $\tilde{V} \subset U_{\tilde{j}}$, and from a diffeomorphism $\tilde{f}\colon \tilde{U}' \to \tilde{V}'$ such that $\phi_{\tilde{j}} \circ \tilde{f} = \phi_{\tilde{i}}$.

If $i \neq \tilde{i}$ or $j \neq \tilde{j}$ then $\psi(U')$ is disjoint from $\tilde{\psi}(\tilde{U}')$ and there is nothing to prove.

Suppose that $i = \tilde{i}$ and $j = \tilde{j}$, so that both $U'$ and $\tilde{U}'$ are open subsets of $U_i$. Then we have the following lemma:

5.6. **Lemma.** *The set*

$$\Delta_{f\,\tilde{f}} = \{x \in U' \cap \tilde{U}' \mid \operatorname{germ}_x f = \operatorname{germ}_x \tilde{f}\}$$

*is a union of connected components of $U' \cap \tilde{U}'$ (ie, it is both open and closed).*

*Proof.* The openness of $\Delta_{f\,\tilde{f}}$ is a consequence of a the definition of $\operatorname{germ}_x f$: if $\operatorname{germ}_x f = \operatorname{germ}_x \tilde{f}$, the same is true for points in a neighbourhood of $x$.

For closedness of $\Delta_{f\,\tilde{f}}$, supposed that there is a point $\hat{x} \in U' \cap \tilde{U}'$ such that $\hat{x} \in \partial\Delta_{f\,\tilde{f}} \setminus \Delta_{f\,\tilde{f}}$: we will use the fact that $X$ is a diffeological orbifold to linearize the action of $G_{\hat{x}}$ (the stabilizer of the image of $\hat{x}$ in $X$) in a neighbourhood of $\hat{x}$ and conclude that $\operatorname{germ}_{\hat{x}} f = \operatorname{germ}_{\hat{x}} \tilde{f}$, contradicting the assumption that $x \notin \Delta_{f\,\tilde{f}}$.

To do so, choose a neighbourhood $U_{\hat{x}} \cong \mathbb{R}^n$ of $\hat{x}$ in $U' \cap \tilde{U}'$ small enough such that $\phi_i\big|_{U_{\hat{x}}}$ is isomorphic to the quotient map $U_{\hat{x}} \to U_{\hat{x}}/G_{\hat{x}}$ for some linear action of $G_{\hat{x}}$ on $U_{\hat{x}}$; also, shrink $U_{\hat{x}}$ if needed, to make sure the inclusion $V_{\hat{x}} := f(U_{\hat{x}}) \subseteq \tilde{V}'$ holds true.

Now, since $f(\hat{x}) = \tilde{f}(\hat{x})$ (by continuity) there is a small neighbourhood of $\hat{x}$ (which we still denote by $U_{\hat{x}}$) on with the diffeomorphism $f^{-1} \circ \tilde{f}$ is an orbit preserving diffeomorphism, and so by Lemma 5.2 it is a can be identified with an element of the group $G_{\hat{x}}$. But, this group's action is linear, and so we conclude that $f^{-1} \circ \tilde{f}$ acts as the identity map because by assumption it is identity on an open subset of this smaller open set: more precisely, it is identity on $U_{\hat{x}} \cap \Delta_{f\,\tilde{f}}$ which is nonempty because, by assumption $\hat{x} \in \partial\Delta_{f\,\tilde{f}}$.

Therefore, $\hat{x} \in \Delta_{f\,\tilde{f}}$, and so $\Delta_{f\,\tilde{f}}$ is closed. This together with openness, gives us the desired result that $\Delta_{f\,\tilde{f}}$ is both open and closed, and so a union of connected components of $U' \cap \tilde{U}'$.

□

Now consider the maps

$$\psi\colon U' \to \Gamma(\mathcal{U})_1$$

and

$$\tilde{\psi}\colon \tilde{U}' \to \Gamma(\mathcal{U})_1.$$

Then $W := \psi^{-1}\tilde{\psi}(\tilde{U}')$ is exactly the set of Lemma 5.6; in particular, this set is open in $U'$. The map $\tilde{\psi}^{-1} \circ \psi$ is the inclusion map of $W$ in $\tilde{U}'$; in particular, this map is smooth.

Thus, there exists a unique topology on $\Gamma(\mathcal{U})_1$ such that the maps $\psi$ are homeomorphisms with open subsets of $\Gamma(\mathcal{U})_1$, and, moreover, these maps form an atlas on $\Gamma(\mathcal{U})_1$.

□

5.7. **Lemma.** *Let $K$ be a finite subgroup of* $\mathrm{GL}_n(\mathbb{R})$*, and let $U \subset \mathbb{R}^n$ be a $K$-invariant open subset. Let $\Gamma(U)$ be the groupoid whose set of objects is $U$ and in which an arrow from $x$ to $y$ is the germ at $x$ of a diffeomorphism from a neighborhood of $x$ in $U$ to a neighborhood of $y$ in $U$ that takes $x$ to $y$ and that takes each point to a point in the same orbit.*

*Then the map of groupoids $K \ltimes U \to \Gamma(U)$ given by $(\gamma, x) \mapsto \mathrm{germ}_x \gamma$ is a bijection.*

*Proof.* Suppose $\mathrm{germ}_x \gamma = \mathrm{germ}_x \gamma'$. Because $\gamma$ and $\gamma'$ are analytic, this implies that $\gamma = \gamma'$. Thus, the map $(\gamma, x) \mapsto \mathrm{germ}_x \gamma$ is one-to-one.

Let $x$ and $y$ be points of $U$. Let $f \colon U' \to V'$ be a diffeomorphism between open subsets of $U$ that takes $x$ to $y$ and that takes each point to a point in the same $K$-orbit. After possibly shrinking $U'$, assume that $U'$ is connected. By Lemma 5.2, there exists an element $\gamma$ of $\Gamma$ such that $f$ is equal to the restriction of $\gamma$ to $U'$. Then $\mathrm{germ}_x f = \mathrm{germ}_x \gamma$. Thus, $(\gamma, x) \mapsto \mathrm{germ}_x \gamma$ is onto. □

With the manifold structure described above, the groupoid $\Gamma(\mathcal{U})$ becomes a Lie groupoid. By Lemma 5.7, it is an orbifold groupoid.

The map

$$\bigsqcup_{i \in I} U_i \to \bigsqcup_{i \in I} V_i$$

that takes the element $x$ of $U_i$ to the element $\phi_i(x)$ of $V_i$ induces a diffeomorphism from $\bigsqcup_{i \in I} U_i/G_i$ to $\bigsqcup_{i \in I} V_i$.

By the definition of arrows in the germ groupoid, this map further descends to a map from $\Gamma(\mathcal{U})_0/\Gamma(\mathcal{U})_1$ to $X$. By the definition of the quotient diffeology, this last map is differentiable. We need to show that it is one-to-one and that its inverse is differentiable.

see below.

5.8. **Proposition.** *Given $X$, a diffeological orbifold and $U$ the orbifold atlas constructed above, we have $\Gamma(\mathcal{U})_0/\Gamma(\mathcal{U})_1 \cong X$.*

*Proof.* Recall that the composite map

$$\phi : \Gamma(\mathcal{U})_0 = \coprod_{i \in I} U_i \to \coprod_{i \in I} U_i/G_i = \coprod_{i \in I} V_i \to X,$$

was introduced in the above construction. The claim is that this map descends to a map $\overline{\phi} \colon \Gamma(\mathcal{U})_0/\Gamma(\mathcal{U})_1 \to X$, which is a diffeomorphism of diffeological spaces.

Before proceeding to the smooth structure, we need to answer some set-theoretic questions, which are addressed in the following:

a) (Well-definedness) This is simply because, by the commutative diagram in the definition of $\Gamma(\mathcal{U})_1$, if two points $p, q, \in \Gamma(\mathcal{U})_0$ lie in the same $\Gamma(\mathcal{U})$ orbit, then they map to the same point in $X$, or in other words, $\phi(p) = \phi(q)$.

b) (Injectivity) sharpen this: To show injectivity, we need to take two points $q_1 \in U_i$ and $q_2 \in U_j$ that go down to the same point in $X$ and construct a local diffeomorphism between a neighbourhood of $q_1$ and a neighbourhood of $q_2$: this would mean that $q_1$ and $q_2$ lie in the same $\Gamma(\mathcal{U})$-orbit.

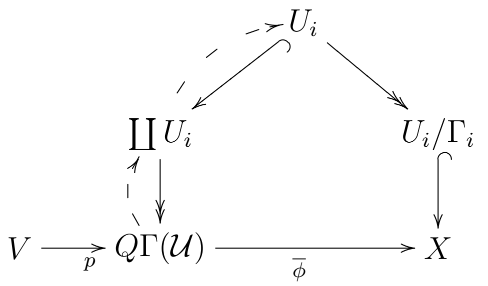
To construct this local diffeomorphism, we make use of Lemma 5.3 by taking a neighbourhood of $q_1$ that is equivariantly diffeomorphic to $\mathbb{R}^n$ and maps into the image of $U_j$ in $X$ under $\phi$. In that case, the conditions of Lemma 5.3 are satisfied and so by that lemma, we have a map from this smaller neighbourhood of $q_1$ to $U_j$ that commutes with the morphisms to $X$.

c) (Surjectivity) This is immediate because of the fact that the images of the $U_i$'s cover $X$.

Now, we need to show that both the map $\overline{\phi}$ and its inverse are differentiable. This is also not too difficult because we can use the locality axiom for diffeologies (cf. Definition 2.1) to restrict attention to a single chart, as in the following:

a) (Differentiability of $\overline{\phi}$) Given a plot $p\colon V \to Q\Gamma(\mathcal{U})$ and a point $x \in V$, by the definition of the quotient diffeology given in Def. 2.4, we can first lift the plot locally to $q^x : W^x \to \Gamma(\mathcal{U})_0 \coprod_{i\in I} U_i$ and then break it up into smaller charts $q_i^x\colon W_i^x \to U_i$, where each $W_i^x = q^{-1}(U_i)$. Then we can compose with the map $U_i \twoheadrightarrow U_i/\Gamma_i \hookrightarrow X$ to get a plot of $X$.

   Now, since the codomains of the plots $\{W_i^x \xrightarrow{q_i^x} U_i \to X \mid x \in V,\ i \in I\}$ cover the image of $V$ in $X$, we have that $V \xrightarrow{p} Q\Gamma(\mathcal{U}) \xrightarrow{\overline{\phi}} X$ is a plot by the locality axiom in Def. 2.1 of diffeological spaces. Here is the diagram that describes this piece of the argument:




b) (Differentiability of $\overline{\phi}^{-1}$) Looking at the above diagram, one realizes that the argument in the previous step can be carried out in the opposite direction from $X$ to $Q\Gamma(\mathcal{U})$, in the sense that one can shrink the plot, so that it would fit inside the image of an orbifold chart $U_i/\Gamma_i \subseteq X$, and then lift through the projection $U_i \twoheadrightarrow U_i/\Gamma_i$, and then compose down to $Q\Gamma(\mathcal{U})$.

□

## 6. Effective étale Lie groupoids

Our goal is to show that the diffeological approach to orbifolds is equivalent to the approach through groupoids. However, there is no chance of obtaining such a result without assuming some version of "effectiveness". clarify why! Diffeology can at most capture the "effective part" of an orbifold.

Let us first introduce the following notation. Let $G = \begin{array}{c} G_1 \\ {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} \\ G_0 \end{array}$ be a Lie groupoid. For open subsets $U$ and $V$ of $G_0$, let

$$\mathrm{Diff}^Q(U,V) = \{\text{diffeomorphisms } \phi\colon U \to V \mid \text{ the triangle } \begin{array}{ccc} U & \xrightarrow{\ \phi\ } & V \\ & \searrow \quad \swarrow & \\ & G_0/G_1 & \end{array} \text{ commutes .}\}$$

6.1. **Definition.** Given an Lie groupoid $G = \begin{array}{c} G_1 \\ {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} \\ G_0 \end{array}$, its **germ groupoid**, $\Gamma(G)$, is the Lie groupoid whose objects are $\Gamma(G)_0 = G_0$, whose arrows are

$$\Gamma(G)_1 = \{\mathrm{germ}_x \phi \mid \phi \in \mathrm{Diff}^Q(U,V) \text{ and } x \in U\},$$

and whose structure maps are the obvious ones.

6.2. *Remark.* We defined $\Gamma(G)$ as an abstract groupoid, not as a Lie groupoid. Later we will see that if $G$ is an orbifold groupoid then $\Gamma(G)$ is naturally a Lie groupoid.

6.3. **Definition.** A Lie groupoid $G = \begin{array}{c} G_1 \\ {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} \\ G_0 \end{array}$ is called **étale** if the source map $s$ (and hence the target map $t$) is a local diffeomorphism (i.e., every point in $G_1$ has a neighbourhood which is mapped by $s$ diffeomorphically onto an open subset of $G_0$).

6.4. *Remark.* Because the inverse map $i\colon G_1 \to G_1$ is an involution that switches the source map with the target map, the source map is a local diffeomorphism if and only if the target map is a local diffeomorphism.

Black Box:

6.5. *Example.* An orbifold groupoid is étale.

check if this is correct.

An étale groupoid has a natural map to its germ groupoid:

6.6. **Definition.** Given an étale groupoid $G$, and points $p, q \in G_0$, we define the map $G_{p,q} \to \Gamma(G)_{p,q}$ as follows: given $g \in G_{p,q}$, choose a neighbourhood $U$ of $g$ such that $s|_U$ and $t|_U$ are diffeomorphisms; now, send $g$ to the germ of the diffeomorphism

$$t|_U \circ (s|_U)^{-1} : s(U) \to t(U)$$

at the point $s(g) = p$.

6.7. *Remark.* This map does not depend on the choice of the open set $U$, since the maps $s$ and $t$ are local diffeomorphisms.

We are now ready to define an effective étale groupoid.

6.8. **Definition.** Let $G$ be an étale Lie groupoid, let $\Gamma(G)$ be its germ groupoid, and let $G \to \Gamma(G)$ be the natural homomorphism defined above. The étale groupoid $G$ is called **effective** if the map $G_1 \to \Gamma(G)_1$ is injective.

Example: a Lie groupoid is étale if and only if it is locally étale. (IS THIS TRUE???) The action groupoid of a discrete group is étale. Thus, every orbifold groupoid is étale.

6.9. *Remark.* An étale groupoid $G$ is effective if and only if, for every two points $p, q \in G_0$, the map $G_{p,q} \to \Gamma(G)_{p,q}$ is injective.

Black Box:

6.10. *Example.* An orbifold groupoid is effective if and only if the finite group actions in its description are effective. need to word better.

The next step is to restrict further to the case of orbifold groupoids and show that in this case the map $G \to \Gamma(G)$ is a bijection, which allows us to put a Lie groupoid structure on $\Gamma(G)$. This is the content of the next lemma: previously stated for effective proper étale.

6.11. **Lemma.** *Given an effective proper étale groupoid G and points $p, q \in G_0$, the injection $G_{p,q} \hookrightarrow \Gamma(G)_{p,q}$, provided by definition 6.8 is in fact a bijection, so*

$$G_{p,q} \cong \Gamma(G)_{p,q}.$$

*Proof.* Note that all we need to show here is that the map $G \to \Gamma(G)$ is onto, since injectivity is guaranteed by the effectiveness assumption. To this end, we pick two arbitrary points $p, q \in G_0$, and open neighbourhoods $U_p$ and $U_q$ of $p$ and $q$ given by Definition 4.3 above, which are diffeomorphic to an open balls in $\mathbb{R}^n$, choosing coordinates such that the groups $G_p$ and $G_q$ act linearly on $U_p$ and $U_q$, respectively.

First, we show the assertion of the lemma in the case when $p = q$, and then extend it to the general case. If $p = q$, we can just use Lemma 5.2 to show that the map $G_p \to \Gamma(G)_p$ is onto: this is because the Lemma tells us that every orbit preserving diffeomorphism comes from an element of the group.

In the situation when $p \neq q$, we use the following observation: for any groupoid $K$, there is a one to one correspondence between the three sets $K_p$, $K_q$, and $K_{p,q}$. In fact, there is a group isomorphism $K_p \cong K_q$ given by fixing an element $k \in K_{p,q}$:

$$\begin{aligned} c_k \colon K_p &\longrightarrow K_q \\ g &\longmapsto kgk^{-1} \end{aligned},$$

which can easily be seen to be bijective exactly because we have cancelation law in groupoids. Similarly, $K_{p,q}$ is a $K_p \cong K_q$ torsor, since after fixing an element $k \in G_{p,q}$, we have the following bijections:

$$\begin{array}{ccccc} K_p & \xrightarrow{\cong} & K_q & \longrightarrow & K_{p,q} \\ g & \longmapsto & h & \longmapsto & kg = hk, \end{array}$$

where $h = c_k(g) = kgk^{-1}$ is as defined above.

So, this shows that $G_{p,q}$ and $\Gamma(G)_{p,q}$ have the same number of element (since $G_{p,q}$ and $\Gamma(G)_{p,q}$ are in bijection with $G_p$ and $\Gamma(G)_p$, and we have seen above that $G_p \cong \Gamma(G)_p$). Now, since we know that the map $G_{p,q} \to \Gamma(G)_{p,q}$ is injective, this gives us that it is also onto. $\square$

Put where it's used:

Proper étale groupoids have very simple local structure, viz. they are locally isomorphic to the action groupoid of a finite group action on an open subset of $\mathbb{R}^n$. This is a very crucial fact that we will be using over and over again in the following sections. But, first we need to introduce an auxiliary definition that will be used when we define an effective étale groupoid.

6.12. **Definition.** A Lie groupoid $G = \begin{smallmatrix} G_1 \\ s \downdownarrows t \\ G_0 \end{smallmatrix}$ is **proper** if the map $s \times t \colon G_1 \to G_0 \times G_0$ is proper (i.e., the preimages of compact sets are compact).

6.13. *Remark.* (i) The terminology *proper groupoid* is motivated by the special case of action groupoids. An action groupoid is proper if and only if it is associated to a proper Lie group action. Indeed, the definition for a group action to be proper is exactly the condition that the map $(g, m) \mapsto (m, g \cdot m)$, from $G \times M \to M \times M$, be proper; this map is exactly the map $s \times t = pr_M \times \mu$.

## 7. An effective proper étale Lie groupoid is determined up to Morita equivalence by its underlying diffeological orbifold

In section 4 we obtained a functor from orbifold Lie groupoids to diffeological orbifolds, which by section 3 sends Mortia equivalences to diffeomorphisms. In section 5 we showed that this functor is essentially surjective.

What we are going to show here is that if the quotients of two proper étale groupoids are isomorphic, then the original groupoids must be Morita equivalent, i.e.,

$$QG \cong QH \Rightarrow G \simeq H.$$

This establishes a bijection between Morita equivalence classes of effective orbifold groupoids and diffeomorphism classes of diffeological orbifolds.

The plan of attack here is to find a third groupoid which has $X := QG \cong QH$ as its quotient and which is Morita equivalent to each one of $G$ and $H$; the picture is as follows:

$$\begin{array}{ccccccc} G & & \simeq & K & \simeq & & H \\ & \searrow & \swarrow & & \searrow & \swarrow & \\ & & QG & \cong & QH & & \end{array}$$

$K$ can be thought of as a refinement of the two orbifold structures on $X$ arising from $G$ and $H$.

7.1. *Construction.* The construction of $K$ is carried out in five steps:

Step 1: We start by using Definition 4.3 to each point $q \in H_0$ and choosing a connected open $H_q$-invariant neighbourhood $V_q$ of $q$ such that $V_q$ is diffeomorphic $\mathbb{R}^n$ and $H|_{V_q} \cong H_q \ltimes V_q$; this gives us a suitable open cover $\mathcal{V}$ of $H_0$.

Step 2: Now, what we can do is to project this open cover down to $X := QH \cong QG$ to get an open cover of $X$ (recall that quotient morphisms are open), which we denote by $\overline{\mathcal{V}}$. We use this cover to choose the open cover on $G$, which will be used to construct $K$.

Step 3: For each point $x \in X$, we pick points $p \in G_0$ and $q \in H_0$ (sometimes denotes as $p_x$ and $q_x$), that map to it under the quotient maps from $G_0$ and $H_0$ to $X$. Then, using Definition 4.3, we pick an open $G_p$-invariant neighbourhood $U_p$ of $p$, which is diffeomorphic to $\mathbb{R}^n$ and maps into $\overline{V}_q$, the image of $V_q$ in $X$. Then, to simplify matters later on, what we do is to shrink the neighbourhoods $V_q \subseteq H_0$ until they map onto the same neighbourhood $W_x$ of $x$. At this point, the relevant picture can be seen Fig. 2.

FIGURE 2. The construction of the Morita equivalence

Step 4: Now, we have an open cover of $X$ by these open sets $W_x$, which can be thought of as quotients of $\mathbb{R}^n$ by a finite linear group (by using a $G_p$-invariant metric to identify $U_p$ with $\mathbb{R}^n$ and $G_p$ with a subgroup of $GL_n(\mathbb{R})$). To keep things manageable, we pick a locally finite subcover $\{W_x\}_{x\in\Delta}$, where $\Delta \subseteq X$ is a necessarily discrete set of points.

Step 5: Now, we define the objects of $K$ as

$$K_0 = \coprod_{x\in\Delta} U_{p_x},$$

and its arrows using the Cartesian square

$$\begin{array}{ccc} K_1 := G_1 \times_{G_0\times G_0} (K_0 \times K_0) & \xrightarrow{\phi_1} & G_1 \\ \downarrow{\scriptstyle (s_K,t_K)} & & \downarrow{\scriptstyle (s,t)} \\ K_0 \times K_0 & \xrightarrow{\phi_0\times\phi_0} & G_0 \times G_0, \end{array}$$

which is the same thing as pulling back the arrows of $G$ to define the groupoid structure on $K$. In this diagram we have introduced the notation $\phi_0$ for the tautological map $K_0 \to G_0$, and $\phi_1 : K_1 \to G_1$ is the map that is given to us by this Cartesian square.

□

The first important fact is that $K$ is an effective orbifold groupoid:

7.2. **Lemma.** *$K$ is effective, proper, and étale and so it is an effective orbifold groupoid.*

*Proof.* Let us address properness first: in order to show that $K$ is proper, we need to prove that the map $(s_K, t_K) : K_1 \to K_0 \times K_0$ (which appeared in the diagram of maps that was used in Step 5 of the above construction to define $K_1$) is a proper map (i.e. the preimage of compact subsets are compact). To do so, pick a compact subset $C \subseteq K_0$: we need to show that $(s_K, t_K)^{-1}(C)$ is compact in $K_1$. This is done by noticing that $(s_K, t_K)^{-1}(C) = C \times_{C^G} (s,t)^{-1}(C^G)$, where $C^G = (\phi \times \phi)(C)$, and all the sets involved in this fibre product are compact, which implies that the fibre product itself is compact.

Now, we'll show that $K$ is étale, which is a consequence of the fact that $\phi : K_0 \to G_0$ is a local diffeomorphism. To show that $K$ is étale, we need to prove that for each $k \in K_1$, there is an open neighbourhood of it $U_k$ such that both $s_K\big|_{U_k}$ and $t_K\big|_{U_k}$ are diffeomorphisms. This neighbourhood is constructed as follows: let $g = \phi_1(k)$, $p_1 = s_K(k)$, and $p_2 = t_K(k)$, and choose a neighbourhood $U_g$ of $g$ small enough such that $s\big|_{U_g}$ and $t\big|_{U_g}$ are diffeomorphisms, and we have $s\big|_{U_g} \subseteq \phi_0(U_1)$ and $t\big|_{U_g} \subseteq \phi_0(U_2)$, where $U_i$ is the connected component of $K_0$ that $p_i$ lies in, for $i = 1, 2$. Since, by the construction of $K$, $\phi_0$ restricted to each connected component of $K_0$ is a diffeomorphism, we get that the set

$$U_k := \Big(\phi_0\big|_{U_1}^{-1}(s(U_g)) \times \phi_0\big|_{U_2}^{-1}(t(U_g))\Big) \times_{s(U_g)\times t(U_g)} U_g$$

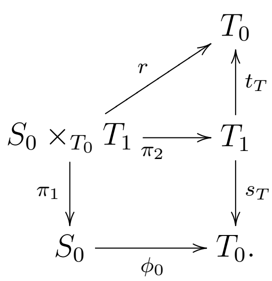
is an open subset of $K_1$, and that $s\big|_{U_k}$ and $t\big|_{U_k}$ are diffeomorphisms: this is because, in the definition of $U_k$ above, we are just pulling back the map $U_g \to s(U_g) \times t(U_g)$ by a diffeomorphism.

Finally, $K$ is effective because $K_0 \to G_0$ is a local diffeomorphism and so for every two points $p, q \in K_0$ with images $p^G$, $q^G$ in $G_0$, we have the bijections $G_{p^G,q^G} \cong K_{p,q}$ and $\Gamma(\mathcal{U}^G)_{p^G,q^G} \cong \Gamma(\mathcal{U}^K)_{p,q}$ (where $\mathcal{U}^G$ and $\mathcal{U}^K$ are open covers of $G_0$ and $K_0$, resp.); therefore, effectiveness of $K$ follows from the effectiveness of $G$. □

The next thing we need is that the tautological map $K \to G$ is a pseudo-equivalence: the (FF) condition holds because of the definition of $K_1$ given above, while (ES) follows from the following lemma (which proves a bit more for later use) and the fact that $K_0 \to G_0$ is a submersion (in fact a local diffeomorphism):

7.3. **Lemma.** *Given a map of groupoids $\phi : S \to T$ such that the map on objects $\phi_0 : S_0 \to T_0$ is a submersion (not necessarily surjective), and $\phi$ is an equivalence of abstract categories, then the (ES) condition of pseudo-equivalence holds. Moreover, if the groupoids are both étale and $\phi_0$ is, in addition, a local diffeomorphism, then (FF) holds as well, and so $\phi$ is a pseudo-equivalence.*

*Proof.* To show the first claim, look at the diagram in the definition of (ES):



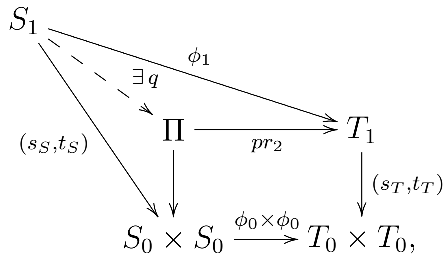
The map $r := t_T \circ \pi_2$ is surjective by the assumption that $\phi$ is an equivalence of categories; so, we just need to show that $r$ is a submersion.

To see this, note that $\phi_0$ is a submersion, so the same is true of its pullback under $s_T$, namely $\pi_2$, which together with the submersiveness of $t_T$ gives that $r = t_T \circ \pi_2$ is a submersion as required.

For the claim about (FF), first of all notice that the submersiveness of $\phi_0$ implies that $\Pi := T_1 \times_{T_0 \times T_0} (S_0 \times S_0)$ in the diagram

$$\begin{CD} \Pi := T_1 \times_{T_0\times T_0} (S_0 \times S_0) @>>> T_1 \\ @VVV @VV(s_T,t_T)V \\ S_0 \times S_0 @>\phi_0\times\phi_0>> T_0 \times T_0. \end{CD}$$

is a manifold. So, we need to show that the map $q : S_1 \to T_1 \times_{T_0\times T_0} (S_0 \times S_0)$ one gets from the universal property of Cartesian squares and the diagram




is a diffeomorphism. However, the condition that $\phi$ is an equivalence of categories tells us that $q$ is a bijection, and so we just need to show that it is a local diffeomorphism, which is where the extra assumptions come in. To show that $q$ is a local diffeomorphism, we will show that both $pr_2$ and $\phi_1$ are local diffeomorphisms, forcing $q$ to be one as well by the commutativity of the diagram.

The fact that $pr_2$ is a local diffeomorphism follows from the fact that it is the pullback along $(s_T, t_T)$ of the local diffeomorphism $\phi_0 \times \phi_0$. As for $\phi_1$, it follows from the fact that all other maps in the following commutative diagram are local diffeomorphisms:

$$\begin{CD} S_1 @>>> T_1 \\ @Vs_SVV @VVs_TV \\ S_0 @>\phi_0>> T_0. \end{CD}$$

Therefore, we get that (FF) holds for $\phi$. □

Now, we will define a map $\psi : K \to H$ based on Construction 7.1 above, and show that it is a pseudo-equivalence. Using Step 3 of the construction and Lemma 5.3, for each $x \in \Delta$, we get a diffeomorphism $U_{p_x} \to V_{q_x}$ which is $G_{p_x} \cong H_{q_x}$-equivariant (where the isomorphism $G_{p_x} \cong H_{q_x}$ is due to Lemma 5.4).

This defines the map $\psi_0 : K_0 \to H_0$ and it defines the map $\psi_1 : K_1 \to H_1$ only on arrows in $K\big|_{U_{p_x}}$, for $x \in \Delta$. To define a the map $\psi_1$ completely, pick points $p_1, p_2 \in K_0$ and denote their image in $H_0$ under the map $\psi_0$ by $p_1^H$ and $p_2^H$, respectively. From the effectiveness of $K$ and Lemma 6.11, we have the bijections $K_{p_1p_2} \cong \Gamma(K)_{p_1p_2}$ and $H_{p_1^H p_2^H} \cong \Gamma(H)_{p_1^H p_2^H}$. On the other hand, the local diffeomorphism $\psi_0$ induces the bijection

$$\begin{aligned} \Gamma(K)_{p_1 p_2} &\xrightarrow{\cong} \Gamma(H)_{p_1^H p_2^H} \\ \operatorname{germ}_{p_1} f &\longmapsto \operatorname{germ}_{p_1^H} \psi_0 \circ f \circ \left(\psi_0\big|_{U_1}\right)^{-1}, \end{aligned}$$

where $U_1$ is the connected component of $K_0$ that $p_1$ lies in, and the composition is defined in a small neighbourhood around $p_1^H$.

Note that this immediately gives us that the map $\psi : K \to H$ is fully faithful (as a map of abstract categories). Moreover, essential surjectivity of $\psi$ (again as a set theoretic functor) is a consequence of the essential surjectivity of the map $K \to G$ because of the isomorphism $G_0/G_1 \cong H_0/H_1$: these two facts guarantee that every orbit of $H_1$ in $H_0$ has an element in the image of $\psi$. In short, $\psi$ gives us an equivalence of categories between the two set theoretic groupoids underlying $K$ and $H$.

Also, note that the map $\psi_0 : K_0 \to H_0$ is a local diffeomorphism, since $\psi_0\big|_{U_{p_x}}$ is a diffeomorphism, as was mentioned while constructing $\psi_0$. So, we can use Lemma 7.3 to conclude that $\psi$ is a pseudo-equivalence, giving us

7.4. **Theorem.** *Given orbifold groupoids* $G = \begin{matrix} G_1 \\ \downarrow\downarrow \\ G_0 \end{matrix}$ *and* $H = \begin{matrix} H_1 \\ \downarrow\downarrow \\ H_0 \end{matrix}$ *and a diffeomorphism of diffeological spaces* $f : G_0/G_1 \cong H_0/H_1$*,* $f$ *can be lifted to a Morita equivalence between* $G$ *and* $H$*, i.e. a third (orbifold) groupoid* $K$ *and pseudo-equivalences* $G \xleftarrow{\sim} K \xrightarrow{\sim} H$*, which induce the map* $f$ *between the quotient spaces.*

## 8. Weak maps and weak pseudo-equivalences

When two manifolds are described by means of gluing coordinate charts, a map between the manifolds can be described in terms of maps from open subsets of the charts of the first manifold to charts of the second manifold. This is the basic example that motivates the definition of weak map.

8.1. **Definition.** Let $G = \begin{array}{c} G_0 \\ {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} \\ G_1 \end{array}$ and $H = \begin{array}{c} H_0 \\ {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} \\ H_1 \end{array}$ be Lie groupoids. Let $\varphi'$ and $\varphi''$ be morphisms from $G$ to $H$. A *smooth natural transformation* from $\varphi'$ to $\varphi''$ is a smooth map

$$\mu\colon G_0 \to H_1$$

that associated to every $x \in G_0$ an arrow $\mu_x$ from $\varphi'(x)$ to $\varphi''(x)$ and such that if $g \in G_1$ is an arrow from $x \in G_0$ to $y \in G_0$ then the following diagram commutes:

$$\begin{array}{ccc} \varphi'(x) & \xrightarrow{\varphi'(g)} & \varphi'(y) \\ \big\downarrow{\scriptstyle \mu_x} & & \big\downarrow{\scriptstyle \mu_y} \\ \varphi''(x) & \xrightarrow{\varphi''(g)} & \varphi''(y) \end{array}$$

We express this in the following diagram.

$$G \underset{\varphi''}{\overset{\varphi'}{\rightrightarrows}} H \qquad \Downarrow \mu$$

8.2. **Lemma.** *Let $G$ and $H$ be Lie groupoids. Let $\varphi'$ and $\varphi''$ be homomorphisms from $G$ to $H$. If there exists a smooth natural transformation from $\varphi'$ to $\varphi''$ then the $\varphi'$ and $\varphi''$ induce the same map from $G_0/G_1$ to $H_0/H_1$.*

*Proof.* Let $x$ be any element of $G_0$, and let $[x]$ be its image in $G_0/G_1$. The map on quotients induced from $\varphi'$ takes $[x]$ to $[\varphi'(x)]$. The map on quotients induced from $\varphi''$ takes $[x]$ to $[\varphi''(x)]$. Because there exists an arrow in $H_1$, namely $\mu_x$, whose source is $\varphi'(x)$ and whose target is $\varphi''(x)$, in the quotient space $H_0/H_1$ the equivalence class $[\varphi'(x)]$ is equal to the equivalence class $[\varphi''(x)]$. □

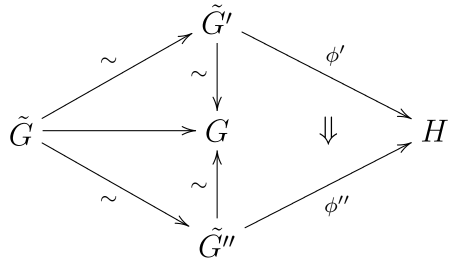
8.3. **Definition.** A *weak map* from a Lie groupoid $G$ to a Lie groupoid $H$ is represented by a pseudo-equivalence $\alpha\colon \tilde{G} \to G$ and a morphism of Lie groupoids $\phi\colon \tilde{G} \to H$. Two such pairs, $(\alpha, \phi')$ and $(\alpha'', \phi'')$, represent the same same weak map if and only if there exists a Lie groupoid $\tilde{G}$, pseudo-equivalences $\beta'\colon \tilde{G} \to \tilde{G}'$ and $\beta''\colon \tilde{G} \to \tilde{G}''$ such that $\alpha' \circ \beta' = \alpha'' \circ \beta''$, and a natural transformation from $\phi' \circ \beta'$ to $\phi'' \circ \beta''$.

Compare with the definition in Moerdijk's paper. Should be the same thing as "inverting" pseudoequivalences.

We express the weak map represented by the pair $(\alpha, \phi)$ by the following diagram.

$$\begin{array}{ccc} \tilde{G} & & \\ {\scriptstyle \sim}\big\downarrow{\scriptstyle \alpha} & \searrow^{\phi} & \\ G & \dashrightarrow & H \end{array}$$

We express two pairs $(\alpha', \phi')$ and $(\alpha'', \phi'')$ that represent the same weak map by the following diagram.



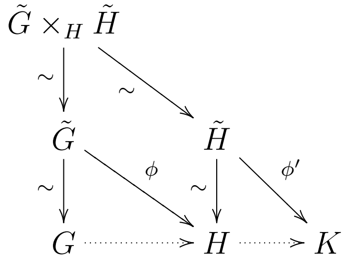

8.4. **Proposition.** *A weak map induces a differentiable map on the quotients as diffeological spaces.*

*Proof.* This follows from Propositions 2.13 and 3.3. □

8.5. *Example.* Let $G$ be the Čech groupoid corresponding to a covering of a manifold $M$, and let $H$ be the Čech groupoid corresponding to a covering of a manifold $N$. A weak map from $G$ to $H$ descends to a smooth map from $M$ to $N$. Conversely, a smooth map from $M$ to $N$ lifts to a unique weak map from $G$ to $H$. need to check the details.

8.6. **Definition.** Let $(\alpha, \phi)$ represent a weak map from $G$ to $H$ and let $(\alpha', \phi')$ represent a weak map from $H$ to $K$. The *composition* of these weak maps is represented by the pair $(\tilde{\alpha}, \tilde{\phi})$ where $\tilde{\alpha}$ is the composition $\tilde{G} \times_H \tilde{H} \to \tilde{G} \to G$ (cf. Definition 3.4) and where $\tilde{\phi}$ is the composition of the natural map from $\tilde{G} \times_H \tilde{H}$ to $\tilde{H}$ with the map $\phi$ from $\tilde{H}$ to $K$.

Thus, we have the following composition diagram.




Composition of weak maps is well defined (independent of the choice of representatives). With this operation, Lie groupoids and weak maps form a category.

We need to understand why this is true; specifically, why considering weak maps is the same as inverting pseudo-equivalences. Eugene explains that natural transformations come up once one insists that composition be associative. He gave a concrete reference that explains this in a more general context of "localization" (which in our case is inverting pseudo-equivalences). Y thinks that the reference was "Handbook of Categorical Algebra", published by Cambridge University Press.

8.7. **Proposition.** *Composition of weak maps descends to composition of the maps on quotient spaces. Thus, we have a functor from the category of Lie groupoids with weak maps to the category of diffeological spaces with differentiable maps.*

*Proof.* need to prove! □

8.8. **Definition.** A weak map $\phi$ is a *weak pseudoequivalence* terminology? if it is invertiable in the category of Lie groupoids and weak maps.

8.9. **Proposition.** *A Morita equivalence between a Lie groupoid $G$ and a Lie groupoid $H$ represents a weak map from $G$ to $H$ and a weak map from $H$ to $G$. These weak maps are inverses of each other. Thus, if two groupoids $G$ and $H$ are Morita equivalent, then there exists a weak pseudoequivalence from $G$ to $H$. Conversely, if there exists a weak pseudoequivalence from $G$ to $H$ then $G$ and $H$ are Morita equivalent.*

*Proof.* Need to prove! □

Maybe insert a comment explaining the role of "2-arrows". We're working with the quotient category of the relevant 2-category.

## 9. The equivalence of categories

There are several different ways to consider orbifold groupoids as a category. We can take the arrows to be Lie groupoid homomorphisms, or weak maps, or weak pseudo-equivalences. The functor from these groupoid categories to diffeological orbifolds is essentially surjective in each of these cases by section 5. Moreover, this remains true for *effective* orbifold Lie groupoids.

We would like to show that the functor from the category of effective orbifold Lie groupoids and weak pseudo-equivalences to the category of diffeological orbifolds and diffeomorphisms is also *fully faithful*.

Let $G$ and $H$ be effective orbifold groupoids. Let $G_0/G_1 \to H_0/H_1$ be a diffeomorphism. We need to show that there exists a unique weak pseudo-equivalence $G \sim H$ that induces this diffeomorphism.

In section 7 we showed that such a weak pseudo-equivalence exists. That is given two effective orbifold groupoids, any diffeomorphism between their quotient spaces as diffeological spaces lifts to a weak pseudo-equivalence of the Lie groupoids. It remains to show that the weak pseudo-equivalence is unique.

We denote these two categories by $\mathcal{EOG}$ and $\mathcal{DO}$.

- Objects of the category $\mathcal{EOG}$ are effective, proper, étale Lie groupoids, $G = \begin{smallmatrix} G_1 \\ s\downarrow\downarrow t \\ G_0 \end{smallmatrix}$.

  Arrows between two such Lie groupoids, $G$ and $G'$, are weak pseudo-equivalences from $G$ to $H$. Thus, an arrow is represented by a pair of pseudo-equivalences
  $$G \overset{\sim}{\leftarrow} K \overset{\sim}{\to} G',$$
  and two such diagrams represent the same arrow if they represent the same weak map in the sense of section 8.
- Objects of the category $\mathcal{DO}$ are diffeological orbifolds. Arrows are diffeomorphisms (in the diffeological sense).

In order to establish the equivalence of these two categories, we need to prove the following proposition

9.1. **Proposition.** *Given a (not necessarily commutative) diagram of pseudo-equivalences*

$$\begin{array}{ccc} & K & \\ \overset{\sim}{\swarrow} & & \overset{\sim}{\searrow} \\ G & & H \\ \underset{\sim}{\nwarrow} & & \underset{\sim}{\nearrow} \\ & K' & \end{array}$$

*such that the induced diagram of diffeomorphisms between the quotient spaces*

$$\begin{array}{ccc} & QK & \\ \overset{\cong}{\nearrow} & & \overset{\cong}{\searrow} \\ QG & & QH \\ \underset{\cong}{\searrow} & & \underset{\cong}{\nearrow} \\ & QK' & \end{array}$$

*is commutative, then $K$ and $K'$ define the same weak pseudo-equivalence, i.e. there is a an effective orbifold groupoid, $L$, making the following diagram 2-commute:*

$$\begin{array}{ccccc} & & K & & \\ & \overset{\sim}{\swarrow} & \uparrow\sim & \overset{\sim}{\searrow} & \\ G & & L & \Downarrow & H \\ & \underset{\sim}{\nwarrow} & \downarrow\sim & \underset{\sim}{\nearrow} & \\ & & K'. & & \end{array}$$

*Proof.* The existence of $L$ and the pseudo-equivalences to $K$ and $K'$ is an immediate consequence of Construction 7.1, and the diffeomorphism $QK \cong QK'$. The fact that the right triangle of the diagram above 2-commutes (by which we mean that there is a natural transformation between the two functors from $L$ to $H$) is a simple ( Masrour: this is intentional; I know this needs more work. The natural transformation should be smooth, but I don't think that's a big problem: one of those lemmas in [IKZ] should do the job. ) consequence of the fact that the diagram of quotient spaces

$$\begin{array}{ccc} QK & & \\ \sim\uparrow & \overset{\cong}{\searrow} & \\ QL & & QH \\ \cong\downarrow & \underset{\cong}{\nearrow} & \\ QK'. & & \end{array}$$

commutes. To make the left triangle commute (as opposed to 2-commute), we will take advantage of the choice involved in constructing the map $L \to K'$ (here we are assuming that the arrows of $L$ are pulled back from $K$ and that the map $L \to K$ is tautological). As in section 7, we can use Lemma 5.3 to construct maps from connected components of $L$ to

$K'$, but each one of these maps can be composed with an element of the group acting on the domain to make sure the left triangle commutes. Here, I probably need to make a reference to the fact that the domains are $\mathbb{R}^n$ and use one of the lemmas out of [IKZ], but I can't see where exactly it would have to come in. □

## References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