MAT 445/1196 - Complex symplectic Lie algebras

Let n be an integer greater than or equal to 2. Let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Then

$$Sp_{2n}(\mathbb{C}) = \{\, g \in GL_{2n}(\mathbb{C}) \mid {}^t gJg = J \,\}$$
$$\mathfrak{sp}_{2n}(\mathbb{C}) = \{\, X \in \mathfrak{gl}_{2n}(\mathbb{C}) \mid {}^t XJ + JX = 0 \,\}.$$

Or, define a nondegenerate bilinear symplectic form on $\mathbb{C}^{2n}$ by $Q(x,y) = {}^t xJy$. Then

$$Sp_{2n}(\mathbb{C}) = \{\, g \in GL_{2n}(\mathbb{C}) \mid Q(gx, gy) = Q(x,y),\ \forall\, x, y \in \mathbb{C}^{2n} \,\}$$
$$\mathfrak{sp}_{2n}(\mathbb{C}) = \{\, X \in \mathfrak{gl}_{2n}(\mathbb{C}) \mid Q(Xx, y) + Q(x, Xy) = 0,\ \forall\, x, y \in \mathbb{C}^{2n} \,\}.$$

We can write elements of $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ in block form: $X = \begin{pmatrix} A & B \\ C & -{}^t A \end{pmatrix}$, where A, B, $C \in M_{n\times n}(\mathbb{C})$ and $B = {}^t B$, $C = {}^t C$. Note that the dimension of $\mathfrak{sp}_{2n}(\mathbb{C})$ is $n(2n+1)$.

The set of diagonal matrices $\mathfrak{h}$ in $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ is an abelian subalgebra of $\mathfrak{g}$. The elements $H_i = E_{i,i} - E_{n+i,n+i}$, $1 \le i \le n$, form a basis of the vector space $\mathfrak{h}$. The subalgebra $\mathfrak{h}$ is a *Cartan* subalgebra of $\mathfrak{g}$.

Let $\{\, \lambda_1, \dots, \lambda_n \,\}$ be the basis of $\mathfrak{h}^*$ that is dual to the basis $\{\, H_1, \dots, H_n \,\}$ of $\mathfrak{h}$: that is, $\lambda_j(H_i) = \delta_{ij}$, $1 \le i, j \le n$.

Consider the adjoint representation $X \mapsto \operatorname{ad} X$ of $\mathfrak{g}$: $\operatorname{ad} X : \mathfrak{g} \to \mathfrak{g}$ is defined by $\operatorname{ad} X(Y) = [X, Y]$, $Y \in \mathfrak{g}$. The set $\operatorname{ad} \mathfrak{h} = \{\, \operatorname{ad} H \mid H \in \mathfrak{h} \,\}$ is a commuting family of semisimple endomorphisms of $\mathfrak{g}$. Hence the operators in $\operatorname{ad} \mathfrak{h}$ are simultaneously diagnonalizable. There exists a finite set $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ of nonzero elements of $\mathfrak{h}^*$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \quad \text{where } \mathfrak{g}_\alpha = \{\, X \in \mathfrak{g} \mid \operatorname{ad} H(X) = [H, X] = \alpha(H) X\ \forall H \in \mathfrak{h} \,\}.$$

This is called the *Cartan decomposition* of $\mathfrak{g}$. Any semisimple Lie algebra has an analogous Cartan decomposition, relative to the restriction of the adjoint representation of $\mathfrak{g}$ to a Cartan subalgebra of $\mathfrak{g}$. The given Cartan subalgebra $\mathfrak{h}$ is always equal to the space $\{X \in \mathfrak{g} \mid [H, X] = 0\, \forall H \in \mathfrak{h} \,\}$. The elements of Φ are called the *roots* of $\mathfrak{g}$ (relative to $\mathfrak{h}$). If $\alpha \in \Phi$, the subspace $\mathfrak{g}_\alpha$ is one-dimensional and is called the *root space* corresponding to α.

Root spaces for $sp_{2n}(\mathbb{C})$:

If $1 \leq i \neq j \leq n$, then $X_{ij} := E_{i,j} - E_{n+j,n+i}$ spans $\mathfrak{g}_{\lambda_i - \lambda_j}$.
If $1 \leq i < j \leq n$, then $Y_{ij} := E_{i,n+j} + E_{j,n+i}$ spans $\mathfrak{g}_{\lambda_i + \lambda_j}$.
If $1 \leq i < j \leq n$, then $Z_{ij} := E_{n+i,j} + E_{n+j,i}$ spans $\mathfrak{g}_{-\lambda_i - \lambda_j}$.
If $1 \leq i \leq n$, then $U_i := E_{i,n+i}$ spans $\mathfrak{g}_{2\lambda_i}$.
If $1 \leq i \leq n$, then $V_i := E_{n+i,i}$ spans $\mathfrak{g}_{-2\lambda_i}$.

Hence the roots for $sp_{2n}(\mathbb{C})$ are

$$\Phi = \{\, \pm(\lambda_i - \lambda_j), \pm(\lambda_i + \lambda_j),\ 1 \leq i < j \leq n;\ \pm 2\lambda_i,\ 1 \leq i \leq n \,\}.$$

In the case of $\mathfrak{sp}_4(\mathbb{C})$, we have $H_1 = \mathrm{diag}(1,0,-1,0)$, $H_2 = \mathrm{diag}(0,1,0,-1)$,

$$X_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad X_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Y_{12} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad Z_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$U_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad U_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Let α, $\beta \in \Phi$. Let $X_\alpha \in \mathfrak{g}_\alpha$ and $X_\beta \in \mathfrak{g}_\beta$. The element $[X_\alpha, X_\beta]$, being an element of $\mathfrak{g}$ has a decomposition as a sum of an element in $\mathfrak{h}$ and some elements in various root spaces. To determine this decomposition, we evaluate $[H, [X_\alpha, X_\beta]]$ for $H \in \mathfrak{h}$. The Jacobi identity tells us that

$$[H, [X_\alpha, X_\beta]] + [X_\alpha, [X_\beta, H]] + [X_\beta, [H, X_\alpha]] = 0.$$

Since $[X_\beta, H] = -[H, X_\beta] = -\beta(H) X_\beta$ and $[H, X_\alpha] = \alpha(H) X_\alpha$, this can be rewritten to get

$$[H, [X_\alpha, X_\beta]] = \beta(H)[X_\alpha, X_\beta] - \alpha(H)[X_\beta, X_\alpha] = (\alpha + \beta)(H)[X_\alpha, X_\beta].$$

It follows that

$$[X_\alpha, X_\beta] \in \begin{cases} \mathfrak{h}, & \text{if } \beta = -\alpha, \\ \mathfrak{g}_{\alpha+\beta}, & \text{if } \alpha+\beta \in \Phi, \\ \{0\}, & \text{otherwise.} \end{cases}.$$

We can see that in the example $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$, we have $\alpha \in \Phi$ if and only if $-\alpha \in \Phi$. This is true in general.

There are certain distinguished subalgebras $\mathfrak{s}_\alpha$ of $\mathfrak{g}$ attached to elements α of Φ. Each of these subalgebras is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Consider the root $\alpha = \lambda_1 - \lambda_2$ of $\mathfrak{sp}_4(\mathbb{C})$. Note that $X_{12} \in \mathfrak{g}_\alpha$, $X_{21} \in \mathfrak{g}_{-\alpha}$, and $[X_{12}, X_{21}] = \operatorname{diag}(1,-1,-1,1) = H_1 - H_2$. We can easily see that the subspace $\mathfrak{s}_\alpha := \operatorname{Span}\{\, H_1 - H_2, X_{12}, X_{21}, \}$ is a subalgebra of $\mathfrak{sp}_4(\mathbb{C})$ that is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. More generally, it is possible to prove that if $\alpha \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$ (hence it is a one-dimensional subspace of $\mathfrak{h}$). Also $[[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha], \mathfrak{g}_\alpha] \neq 0$. These facts can be used to show that $\mathfrak{s}_\alpha := [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is a subalgebra of $\mathfrak{g}$ that is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. In fact, if we let H_α be the unique element of the one-dimensional subspace $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ of $\mathfrak{h}$ such that $\alpha(H_\alpha) = 2$, fixing a nonzero element $X_\alpha \in \mathfrak{g}_\alpha$, we can find a nonzero element $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, Y_\alpha] = H_\alpha$. With these choices, $H_\alpha \mapsto \operatorname{diag}(1,-1)$, $X_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ extends to a Lie algebra isomorphism between $\mathfrak{s}_\alpha$ and $\mathfrak{sl}_2(\mathbb{C})$.

If $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$, set $\alpha = \lambda_1 - \lambda_2$ and $\beta = 2\,\lambda_2$. Then

$$\Phi = \{\, \pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta) \,\}.$$

With this labelling we have

$H_\alpha = H_1 - H_2 = \operatorname{diag}(1,-1,-1,1)$
$H_\beta = H_2 = \operatorname{diag}(0,1,0,-1)$
$H_{\alpha+\beta} = H_1 + H_2 = \operatorname{diag}(1,1,-1,-1)$.
$H_{2\alpha+\beta} = H_1$.

It is immediate from the definition that $H_{-\gamma} = -H_\gamma$ for $\gamma \in \Phi$. When referring to the case $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$, we will reserve the notation α for $\lambda_1 - \lambda_2$. However, when referring to general complex semisimple Lie algebras, α will simply denote any element of Φ.

Let Λ_Φ be the subset of $\mathfrak{h}^*$ made up of all integral linear combinations of elements of Φ, and let $\Lambda_W = \{\, \lambda \in \mathfrak{h}^* \mid \lambda(H_\alpha) \in \mathbb{Z},\ \forall \alpha \in \Phi \,\}$.

Given a root $\alpha \in \Phi$, let w_α be the involution of the vector space $\mathfrak{h}^*$ defined as follows: $w_\alpha(\alpha) = -\alpha$, and $w_\alpha(\lambda) = \lambda$ for all $\lambda \in \Omega_\alpha := \{\, \lambda \in \mathfrak{h}^* \mid \lambda(H_\alpha) = 0 \,\}$. Then

$$w_\alpha(\lambda) = \lambda - (2\lambda(H_\alpha)/\alpha(H_\alpha))\alpha = \lambda - \lambda(H_\alpha)\alpha, \qquad \lambda \in \mathfrak{h}^*.$$

The *Weyl group of* $\mathfrak{g}$ (or of Φ) is defined to be the subgroup of $GL(\mathfrak{h}^*)$ generated by the set $\{\, w_\alpha \mid \alpha \in \Phi \,\}$. Note that it is immediate from the definitions that $w_\alpha = w_{-\alpha}$ for all $\alpha \in \Phi$.

In the case of $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$, the set $\{\, \alpha, \beta \,\}$ is a basis of $\mathfrak{h}^*$ and

$$\begin{aligned}
&\Omega_\alpha = \{\, \lambda \in \mathfrak{h}^* \mid \lambda(H_1) = \lambda(H_2) \,\} = \mathrm{Span}\{\, \alpha + \beta \,\} \\
&\Omega_\beta = \{\, \lambda \in \mathfrak{h}^* \mid \lambda(H_2) = 0 \,\} = \mathrm{Span}\{\, 2\alpha + \beta \,\} \\
&w_\alpha(\beta) = 2\alpha + \beta,\ w_\beta(\alpha) = \alpha + \beta
\end{aligned}$$

It is easy to check that $w_\alpha w_\beta$ has order 4, W is generated by $\{\, w_\alpha, w_\beta \,\}$, and W is isomorphic to the dihedral group of order 8. Furthermore, $W(\Phi) = \Phi$ - this is true in general.

It is possible to show that evey element of W is induced by an automorphism of $\mathfrak{g}$ that carries $\mathfrak{h}$ to itself. In fact, if G is a complex Lie group with Lie algebra $\mathfrak{g}$, and T is the Cartan subgroup of G that corresponds to $\mathfrak{h}$ (that is, T is the closed subgroup of G that is generated by the exponentials of the elements of $\mathfrak{h}$), the group W can be realized as $N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. Given $\alpha \in \Phi$, there is an element $g_\alpha \in N_G(T)$ such that conjugation by g_α induces an automorphism $\mathrm{Ad}\, g_\alpha$ of $\mathfrak{g}$ that preserves $\mathfrak{h}$ and restricts to an automorphism of $\mathfrak{h}$ that corresponds to the automorphism w_α of $\mathfrak{h}^*$.

In the example $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$, T is the group of diagonal matrices in $Sp_4(\mathbb{C})$. For the given choice of $\alpha = \lambda_1 - \lambda_2$, we can take

$$g_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and $\mathrm{Ad}\, g_\alpha(X) = g_\alpha X g_\alpha^{-1}$, $X \in \mathfrak{sp}_4(\mathbb{C})$. Restricting to $\mathfrak{h}$ and then composing, we obtain the involution w_α of $\mathfrak{h}^*$.

Up so multiplication by scalars, there is a unique inner product on $\mathfrak{h}^*$ that is W-invariant. For $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$, since W is generated by w_α and w_β, it suffices to take an inner product on $\mathfrak{h}^*$ that is w_α and w_β-invariant. Denoting the inner product by $\langle \cdot, \cdot \rangle$, we must have α orthogonal to Ω_α, that is, $\langle \alpha, \alpha + \beta \rangle = 0$, and β orthogonal to Ω_β. Hence

$$\langle \alpha, \beta \rangle = -\langle \alpha, \alpha \rangle = -\langle \beta, \beta \rangle / 2.$$

We can (and do) normalize so that α is a unit vector. For convenience, we fix an isometry between our inner product space $\mathfrak{h}^*$ and with the inner product space $\mathbb{C}^2$ (relative to the standard inner product), with α identified with $(1, 0)$. In that case, there are two possible choices for the vector that we identify with β: $(-1, \pm 1)$. We choose to take $(-1, 1)$. Then $\alpha + \beta$ is identified with $(0, 1)$ and $2\alpha + \beta$ with $(1, 1)$.

It is convenient to partition Φ as a disjoint union of two sets Φ^+ and Φ^- in a nice way. One of the properties we need from such a partition is: $\alpha \in \Phi^+$ if and only if $-\alpha \in \Phi^-$. Also, if α and β belong to Φ^+, we require that if $\alpha + \beta$ belongs to Φ, it belongs to Φ^+.

For example, we can choose a (real) linear functional ℓ on the space $\mathrm{Span}_{\mathbb{R}}(\Phi)$ that is nonvanishing on the subset Φ. Then we can set $\Phi^+ = \{ \alpha \in \Phi \mid \ell(\alpha) > 0 \}$ and $\Phi^- = \{ \alpha \in \Phi \mid \ell(\alpha) < 0 \}$. The elements of Φ^+ are referred to as *positive* roots, and the elements of Φ^- are *negative* roots. A choice of Φ^+ (and hence Φ^-) is called an *ordering* on Φ.

In our $\mathfrak{sp}_4(\mathbb{C})$ example, one possible choice for Φ^+ is $\Phi^+ = \{ \alpha, \beta, \alpha + \beta, 2\alpha + \beta \}$.

A positive root is said to be *simple* (or *primitive*) if it cannot be expressed as a sum of two positive roots. (A similar definition can be made for negative roots). For the choice of Φ^+ that we have made for $\mathfrak{sp}_4(\mathbb{C})$, α and β are the simple roots in Φ^+.

In general, suppose that $\Delta = \{ \alpha_1, \ldots, \alpha_\ell \}$ is the set of simple roots in Φ^+. Then Δ is a basis for $\mathfrak{h}^*$ and

$$\Phi^+ \subset \left\{ \sum_{i=1}^{\ell} m_i \alpha_i \mid m_i \in \mathbb{Z},\, m_i \geq 0 \right\}.$$

To be continued....