The Irreducible Representations of A_5

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Contents

1 Introduction

Finding the character table and hence the irreducible characters of a group is not difficult. Realizing these as representations, however, takes a little more work. The aim of this project is to find all the irreducible representations of the group A_5, or rather, at least one representative of each equivalence class. The character table of A_5 is found in Appendix 4.2, and it is:

	e	(123)	$(12)(34)$	(12345)	(13452)
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	5	-1	1	0	0
χ_4	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_5	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

So there are 5 irreducible characters, and in Section 3, representations with each of these characters are found. But first some background theory is needed.

2 Background Theory

Here we list some general results which will be useful later, when looking for representations with specific characters.

2.1 CG-modules

Representations are equivalent to CG-modules, using the following theorem:

Theorem 1:

If $\rho : G \to GL(n, C)$ is a representation of G over C and $V = F^n$ then V becomes a CG-module if we define the multiplication vg by $vg = v\rho(g)$. There is a basis β of V such that $\rho(g) = [g]_\beta$ for all $g \in G$.
Also, if V is a CG-module with basis β then the function $g \mapsto [g]_\beta$ is a representation of G.

Proof:
See [1] p. 40.

Hence we can work with modules rather than with representations, because by Theorem 1 they are equivalent. In particular, a 1 dimensional character

χ (which is a representation), corresponds to a module as follows:

$$U_\chi = span\{\sum_{g\in G} \chi(g)g\}$$

In the search for irreducible modules, we will encounter many which are non-irreducible. It is therefore important to be able to decompose a module into irreducible components. Applying the following theorem allows us to do so.

Maschke's Theorem:

Let G be a finite group and let V be a CG-module. If U is a CG-submodule of V, then there is a CG-submodule W of V such that $V = U \oplus W$.

Proof:
(See [1].) First choose any vector space W' with the desired property, i.e. $V = U \oplus W'$. Then use this to find a module as follows. Define $\phi : V \to V$ by $\phi(v) = u$ where u is the unique element of U such that $v = u + w$ for some element w in W'. I.e. ϕ is the projection of V onto W'. Use ϕ to define $\varphi : V \to V$ by

$$\varphi(v) = \frac{1}{|G|}\sum_{g\in G} g^{-1}(\phi(g(v)))$$

where $g(v)$ means multiplication by g as in Theorem 1. Then φ is linear, and maps V into U, because $y := \phi(x) \in U \quad \forall x \in V$ and since U is a module, $g^{-1}(y) \in U \quad \forall y \in U$. The aim is to show φ is a projection onto U, and let $W = \ker\varphi$. φ is a CG-homomorphism because for all $v \in V, h \in G$ we have

$$\begin{aligned}\varphi(hv) &= \frac{1}{|G|}\sum_{g\in G} g^{-1}(\phi(g(hv))) = \frac{1}{|G|}\sum_{gh\in G} h(gh)^{-1}(\phi((gh)(v))) = \\ &= h\left(\frac{1}{|G|}\sum_{gh\in G} h(gh)^{-1}(\phi((gh)(v)))\right) = h\varphi(v)\end{aligned}$$

So φ is a CG-homomorphism. Also, $\varphi^2 = \varphi$. This is because for all $u \in U, g \in G$ we have $g(u) \in U$ since U is a module. Therefore $\phi(g(u)) = g(u)$ so $g^{-1}\phi(g(u)) = g^{-1}g(u) = u$. Hence

$$\varphi(u) = \frac{1}{|G|}\sum_{g\in G} g^{-1}(\phi(g(u))) = \frac{1}{|G|}\sum_{g\in G} u = u$$

So φ fixes U, hence $\varphi^2 = \varphi$. This also shows $\mathrm{im}\varphi = U$. So φ is a projection onto U and it is a CG-homomorphism. Let $W = \ker\varphi$. Then W is a CG-module since it is the kernel of a CG-homomorphism, and $V = U \oplus W$, as desired. QED

The proof of this will be helpful later.

2.2 Induced Representations

Section 2.1 was concerned with CG-modules in general. In this section we introduce a few facts about a specific way of finding representations and modules; induced representations. This is a powerful way of finding representations of G by looking at subgroups of G.

Definition:

If H is a subgroup of G, and U is a CH-submodule of CH, then let $U \uparrow G = span\{ug : u \in U, g \in G\}$. Call $U \uparrow G$ the *module of G induced from U*. (This is indeed a CG-submodule of CG.) The degree of $U \uparrow G$ is:

$$\deg U \uparrow G = \frac{|G|}{|H|}\chi(e)$$

When inducing modules from a subgroup to G there is often some particular module one aims to find. Therefore it would be helpful to know which module(s) of G a particular module of H will induce to.
In order to find out how a module U with character ψ_k will induce to a module of G, it is helpful to form the matrix A with ij-entry $< \chi_i \downarrow H, \psi_j >_H$. By Frobenius Reciprocity Theorem (proved in class),

$$< \chi_i \downarrow H, \psi_j >_H = < \chi_i, \psi_j \uparrow G >_G$$

Hence, if ψ_k is a character of H, the exact combination of characters of G which ψ_k induces too, is given by the columns of the matrix A. This will be very useful in suggesting beforehand which characters/modules would be useful to induce.

3 Realising the Representations of A_5

The aim of this project is, as mentioned in the introduction, to find at least one representation for each character in the character table of A_5. There are 5 irreducible characters, and they are found in Appendix 4.2:

	e	(123)	$(12)(34)$	(12345)	(13452)
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	5	-1	1	0	0
χ_4	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_5	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

In the remainder of Section 3, explicit irreducible representations with the above irreducible characters are found. In some cases, alternative approaches are given. For consistency purposes, the trivial character has its own section as well as each of the other characters.

3.1 Representation Corresponding to χ_1

The trivial character is one dimensional, so is the trace of a representation into 1×1 matrices, which is the 1×1 matrix itself. So the trivial character equals the trivial representation, which is irreducible.

The other representations will take substantially much more work.

3.2 Representation Corresponding to χ_2

It is shown when finding the character table of A_5 in Appendix 4.2 that $\chi_2(g) = |fix(g)| - 1 \quad \forall g \in G$. This is related to the "*permutation representation*" having character $\chi(g) = |fix(g)| \quad \forall g \in G$. The permutation representation sends g to a matrix permuting the columns in the same way g permutes $\{1, 2, 3, 4, 5\}$. E.g.

$$(123) \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Unfortunately this representation is not irreducible as it has character $\chi = \chi_1+\chi_2$. But we can decompose it into irreducible components using Maschke's Theorem.

Let V be the module equivalent to this representation, and let $\beta = \{v_1, ..., v_5\}$ be a basis. g acts on v_i by $g(v_i) = v_{g(i)}$. Then $U_1 := span\{v_1 + ... + v_5\}$ is a 1-dimensional submodule of V. G acts on U_1 by $g(v_i) = v_g(i) \ \ \forall g \in G$. Therefore $g(u) = u \ \ \forall u \in U_1$. Hence U_1 is equivalent to the trivial module, and thus the character of U_1 is equal to the trivial character.

By Maschke's Theorem U_1 has a complement, i.e. there is a CG-module W such that

$$V = U_1 \oplus W$$

To find W, let $W' = span\{v_1, v_2, v_3, v_4\}$. Then W' is a subspace of V such that $V = U_1 \oplus W'$. Now use this W' to find a CG-module, as in the proof of Maschke's Theorem.
Define

$$\phi : V \to V, \quad \phi : \begin{array}{lll} v_i & \mapsto & 0 \qquad\qquad\qquad i = 1, 2, 3, 4 \\ v_5 & \mapsto & v_1 + ... + v_5 \end{array}$$

i.e. ϕ is the projection onto U_1 with kernel W'. Also define

$$\varphi(v) = \frac{1}{60} \sum_{g \in G} g^{-1} \phi(g(v))$$

Then if $a_i =$ the number of elements which send v_i to v_5, we have

$$\varphi(v_i) = \frac{1}{60} a_i (v_1 + ... + v_5) = \frac{1}{5}(v_1 + ... + v_5)$$

because $a_i = 12 \ \forall i$. By Maschke's Theorem, the W we are looking for is the kernel of φ:

$$W = \ker \varphi = span\{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4 - v_5\} = \{\sum_i \lambda_i v_i | \sum_i \lambda_i = 0\}$$

The representation π_2 equivalent to the module W can be found using Theorem 1, taking as our basis $\beta = \{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4 - v_5\}$. A few examples are:

$$\pi_2 : (123) \mapsto \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\pi_2 : (12)(34) \mapsto \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\pi_2 : (12345) \mapsto \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\pi_2 : (13452) \mapsto \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

The character of this representation is $\chi(g) = \chi_V - \chi_{U_1} = |fix(g)| - 1 \forall g \in G$, which is exactly χ_2. The representation is irreducible since its character agrees with χ_2 which is an irreducible character.

NOTE: The above can be done for S_n or any subgroup of S_n.

3.3 Representation Corresponding to χ_3

The representation corresponding to χ_3 can be found using representations of a subgroup of A_5 and inducing them to representations of A_5. The size of the subgroup together with the degree of the representation determines the degree of the induced representation, as mentioned in Section 2.2:

$$dim(U \uparrow A_5) = \frac{|A_5|}{H} \chi(e)$$

where H is the subgroup of A_5 having a module U with character χ, and $U \uparrow A_5$ is the module of A_5 induced from the module U of H.

A_4 is a subgroup of A_5 of size 12, so a 1-dimensional representation of A_4 would induce to a 5-dimensional representation of A_5. The character table of A_4 (see Appendix 4.3) is as follows:

	e	$(12)(34)$	(123)	(132)
ψ_1	1	1	1	1
ψ_2	3	-1	0	0
ψ_3	1	1	λ	λ^2
ψ_4	1	1	λ^2	λ

where λ is a 3rd root of unity.
So inducing e.g. ψ_3 would give a 5-dimensional representation of A_5, which would hopefully be the desired one, i.e. have character χ_3. In order to see in advance how it will induce, compute the matrix A mentioned in Section 2.2:

	ψ_1	ψ_2	ψ_3	ψ_4
χ_1	1	0	0	0
χ_2	1	1	0	0
χ_3	0	1	1	1
χ_4	0	1	0	0
χ_5	0	1	0	0

So inducing a module of either ψ_3 or ψ_4 will give the required module. Let's choose ψ_3. Since the character ψ_3 is 1-dimensional, it is also a representation. A 1-dimensional CH-module can be created using ψ_3 as follows:

$$U_2 = span\{\sum_{h \in A_4} \psi_3(h)h\}$$

Then

$$\begin{aligned}
U_2 \uparrow A_5 &= span\{ug | u \in U_2, g \in A_5\} = \\
&= span\{\sum_{h \in A_4} \psi_3(h)h, \sum_{h \in A_4} \psi_3(h)h(125), \sum_{h \in A_4} \psi_3(h)h(135), \\
&\quad \sum_{h \in A_4} \psi_3(h)h(145), \sum_{h \in A_4} \psi_3(h)h(152)\} = \\
&= span\{e + (12)(34) + (13)(24) + (14)(23) + \lambda(123) + \lambda(134) + \\
&\quad \lambda(142) + \lambda(243) + \lambda^2(132) + \lambda^2(143) + \lambda^2(234) + \lambda^2(124), \\
&\quad (125) + (25)(34) + (14253) + (13254) + \lambda(13)(25) + \lambda(12534) + \\
&\quad \lambda(254) + \lambda(14325) + \lambda^2(253) + \lambda^2(12543) + \lambda^2(13425) + \lambda^2(14)(25), \\
&\quad (135) + (14352) + (24)(35) + (12354) + \lambda(235) + \lambda(14)(35) +
\end{aligned}$$

$$\lambda(13542) + \lambda(12435) + \lambda^2(12)(35) + \lambda^2(354) + \lambda^2(14235) + \lambda^2(13524),$$
$$(145) + (13452) + (12453) + (23)(45) + \lambda(14523) + \lambda(345) +$$
$$\lambda(12)(45) + \lambda(13245) + \lambda^2(14532) + \lambda^2(13)(45) + \lambda^2(12345) + \lambda^2(245),$$
$$(152) + (15)(34) + (15423) + (15324) + \lambda(153) + \lambda(15234) +$$
$$\lambda(15)(24) + \lambda(15432) + \lambda^2(15)(23) + \lambda^2(15243) + \lambda^2(15342) + \lambda^2(154)\}$$

This has dimension 5 so let these 5 elements of be a basis β of $U_2 \uparrow A_5$. That makes $U_2 \uparrow A_5$ equivalent to the representation

$$\pi_3 : g \mapsto [g]_\beta$$

as in Theorem 1. Here are a few examples of how π_3 acts:

$$\pi_3 : (123) \mapsto \begin{pmatrix} \lambda^2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & \lambda & 0 & 0 & 0 \end{pmatrix}$$

$$\pi_3 : (12)(34) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\pi_3 : (12345) \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & \lambda \\ 0 & 0 & \lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 & 0 \end{pmatrix}$$

$$\pi_3 : (13452) \mapsto \begin{pmatrix} 0 & \lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & \lambda & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 & 0 \end{pmatrix}$$

Calculating the traces of these (and noting that $\lambda^2 + \lambda + 1 = 0$), we see that the representation π_3 has character χ_3, so must be irreducible.

3.4 Representation Corresponding to χ_4

We could find a representation corresponding to χ_4 using modules in a similar approach to that in Sections 3.2 and 3.3 (see Section 3.4.1). Alternatively, and perhaps more interesting, is the approach taken in Section 3.4.2, using a geometric interpretation of A_5. This also turns out to be much shorter.

3.4.1 Algebraic Approach

If we can find a subgroup of the right size, we can use the same approach as in finding the representation corresponding to χ_3. Unfortunately the subgroups would have to be of size 20 or larger, but the largest subgroup of A_5 is A_4, of size 12. However, a smaller subgroup would have representations which induce to linear combinations of representations of A_5, which could then be decomposed to irreducible components.

The second largest subgroup of A_5 is isomorphic to D_{10}. For example, let

$$D_{10} = \{e, (12345), (13524), (14253), (15432),$$
$$(12)(35), (13)(45), (14)(23), (15)(24), (25)(34)\}$$

The character table of D_{10} is (see Appendix 4.4):

	e	$(12)(35)$	(12345)	(13452)
ϕ_1	1	1	1	1
ϕ_2	1	-1	1	1
ϕ_3	2	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
ϕ_4	2	0	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$

In order to find out beforehand how these representations will induce to representations of A_5, we again calculate the matrix A mentioned earlier in Section 2.2:

	ϕ_1	ϕ_2	ϕ_3	ϕ_4
χ_1	1	0	0	0
χ_2	0	0	1	1
χ_3	1	0	1	1
χ_4	0	1	1	0
χ_5	0	1	0	1

To find a module of χ_4 we can induce a module of ϕ_3, and then break it down into irreducible components. To do that we first need to find a module for ϕ_3. This leads us to look at the subgroup $Z_5 = \{e, (12345), (13524), (14253), (15432)\}$ of D_{10}. It has character table:

	e	(12345)	(13524)	(14253)	(15432)
φ_1	1	1	1	1	1
φ_2	1	μ	μ^2	μ^3	μ^4
φ_3	1	μ^2	μ^3	μ^4	μ
φ_4	1	μ^3	μ^4	μ	μ^2
φ_5	1	μ^4	μ	μ^2	μ^3

where μ is a 5th root of unity.

Using the formula in Section 2.1 a module for e.g φ_2 is $U_3 = span\{e + \mu(12345) + \mu^2(13524) + \mu^3(14253) + \mu^4(15432)\}$. Call this element x, so $U_3 = span\{x\}$. Inducing this to D_{10} we get

$$\begin{aligned} U_3 \uparrow D_{10} &= span\{x, x(12)(35)\} \\ &= span\{e + \mu(12345) + \mu^2(13524) + \mu^3(14253) + \mu^4(15432), \\ &\quad (12)(35) + \mu(13)(45) + \mu^2(14)(23) + \mu^3(15)(24), \mu^4(25)(34)\} \end{aligned}$$

Inducing this further, to a module of A_5, we get:

$$\begin{aligned} (U_3 \uparrow D_{10}) \uparrow A_5 &= span\{x, x(12)(35) \\ &= span\{e + \mu(12345) + \mu^2(13524) + \mu^3(14253) + \mu^4(15432), \\ &\quad (12)(35) + \mu(13)(45) + \mu^2(14)(23) + \mu^3(15)(24), \mu^4(25)(34) \\ &\quad (123) + \mu(13245) + \mu^2(14)(25) + \mu^3(15342) + \mu^4(354), \\ &\quad (124) + \mu(13425) + \mu^2(14352) + \mu^3(153) + \mu^4(23)(45), \\ &\quad (125) + \mu(13452) + \mu^2(14)(35) + \mu^3(15423) + \mu^4(243), \\ &\quad (135) + \mu(14523) + \mu^2(15324) + \mu^3(254) + \mu^4(12)(34), \\ &\quad (145) + \mu(15234) + \mu^2(24)(35) + \mu^3(12543) + \mu^4(132), \\ &\quad (134) + \mu(14235) + \mu^2(15243) + \mu^3(253) + \mu^4(12)(45), \\ &\quad (234) + \mu(12435) + \mu^2(13)(25) + \mu^3(14532) + \mu^4(154), \\ &\quad (235) + \mu(12453) + \mu^2(13254) + \mu^3(142) + \mu^4(15)(34), \\ &\quad (245) + \mu(12534) + \mu^2(13542) + \mu^3(143) + \mu^4(15)(23), \\ &\quad (345) + \mu(12354) + \mu^2(13)(24) + \mu^3(14325) + \mu^4(152), \} \end{aligned}$$

This is *not* an irreducible module; it is a combination of a module with character χ_2, one with character χ_3 and one with character χ_4. We could try to factor out a module with character χ_3 e.g., but since the module found earlier in Section 3.3, may only be *equivalent* to the module to be factored out, this may be very difficult. Instead, we could try to project this 12 dimensional module onto a 3 dimensional submodule, with character χ_4. This involves calculating $\sum_{g \in A_5} \chi_4(g)[g]_\beta$, which is a sum of 40 12×12 matrices. (The 20 3-cycles have $\chi_4(g) = 0$ so they will not appear in the sum.) This will give the required module and representation, although it is very computational.

3.4.2 Geometric Approach

There are many different ways to find representations with a given character. Here is a completely different approach to that above, using geometrical interpretations rather than modules.

The icosahedron, the 3-dimensional platonic solid consisting of 20 equilateral triangles, has symmetry group related to A_5. This fact can be used to find a representation of degree 3 of A_5. The symmetry group of the icosahedron is $A_5 \times Z_2$, and the rotational symmetry group is just A_5 (see [2]). This can be seen as follows: Let an *orthogonal set* be a set of 6 points, so that 3 pairwise orthogonal lines can be drawn between pairs of them. The midpoints of the 30 edges of icosahedron can be divided into 5 orthogonal sets (see Figure 1 below). Once an edge is specified, the entire orthogonal set is specified, so these are permuted by the rotations in the symmetry group. Since there are 5 of them, the rotational symmetry group is isomorphic to A_5.

This shows A_5 is isomorphic to a subgroup of $GL_3(C)$, with an isomorphism sending any element of A_5 to the equivalent rotation of the orthogonal sets. This is exactly the definition of a representation of degree 3. It is not yet clear weather this representation has character χ_4 or χ_5, but because A_5 has no representations of degree 2, it is likely that any 'nontrivial' representation of degree 3 is irreducible.

To see how each element of A_5 corresponds to a rotation, consider the image of an icosahedron in Figure 1. Here the orthogonal sets have been numbered, so that an element g of A_5 corresponds to the rotation permuting the orthogonal sets in the same way g permutes the numbers 1 to 5. Take the coordinate system to have horizontal x-axis, vertical z-axis, and a y-axis perpendicular to the image.

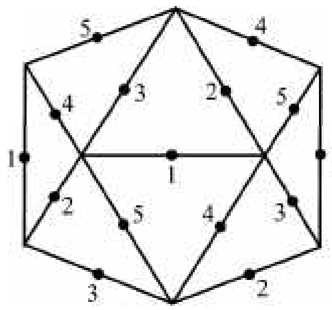

Figure 1: Orthogonal sets in Icosahedron

Define this representation of A_5 as follows:

$$\pi_4 : g \mapsto$$

$$\begin{pmatrix} \cos\theta + (1-\cos\theta)x^2 & (1-\cos\theta)xy - z\sin\theta & (1-\cos\theta)xz + y\sin\theta \\ (1-\cos\theta)xy + z\sin\theta & \cos\theta + (1-\cos\theta)y^2 & (1-\cos\theta)yz - x\sin\theta \\ (1-\cos\theta)xz - y\sin\theta & (1-\cos\theta)yz + x\sin\theta & \cos\theta + (1-\cos\theta)z^2 \end{pmatrix}$$

where g corresponds to the rotation of the icosahedron with axis (x, y, z) (a unit vector) and angle θ. (This is a rotation matrix; see [3].) All axes of rotation corresponding to 5-cycles pass through a pair of vertices of the icosahedron; the reason these have order 5 is that there are 5 faces attached to each vertex. 3-cycles on the other hand correspond to rotations with axes passing through the centre of two opposite faces; the faces are triangles, so these rotations have order 3. Finally, two 2-cycles have rotations with axes passing through the midpoints of two edges, and have order 2.

For example, the element (12345) of A_5 corresponds to a rotation with angle $\frac{2\pi}{5}$ and axis of rotation the unit vector $a(\frac{1}{2}, \frac{1+\sqrt{5}}{4}, 0)$ where $a = \sqrt{\frac{8}{5+\sqrt{5}}}$. Therefore

$$\pi_4 : (12345) \mapsto$$

$$\begin{pmatrix} \cos\frac{2\pi}{5}(1-\frac{a^2}{4}) + \frac{a^2}{4} & (1-\cos\frac{2\pi}{5})a^2\frac{1+\sqrt{5}}{8} & a\frac{1+\sqrt{5}}{4}\sin\frac{2\pi}{5} \\ (1-\cos\frac{2\pi}{5})a^2\frac{1+\sqrt{5}}{8} & \cos\frac{2\pi}{5}\left(1-a^2\frac{3+\sqrt{5}}{2}\right) + a^2\frac{3+\sqrt{5}}{2} & -\frac{1}{2}\sin\frac{2\pi}{5} \\ -a\frac{1+\sqrt{5}}{4}\sin\frac{2\pi}{5} & \frac{a}{2}\sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} \end{pmatrix}$$

To calculate the trace we first need to find the exact value of $\cos\frac{2\pi}{5}$. This can be done by noting that one of the 5th roots of unity is $\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, and the 5th roots of unity can easily be found. The result is that $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$

and $\sin\frac{2\pi}{5} = \sqrt{\frac{5+\sqrt{5}}{8}}$. So the trace is $\frac{1+\sqrt{5}}{2}$. Hence this representation has character χ_4 because no other degree 3 character (irreducible of not) takes this value on (12345).

Similarly one can calculate $\pi_4((13452))$. This rotation has angle $\frac{6\pi}{5}$ and axis of rotation the unit vector $a(-\frac{1+\sqrt{5}}{4}, 0, \frac{1}{2})$ where a is as before[1]. Therefore:

$$\pi_4 : (13452) \mapsto$$

$$\begin{pmatrix} \cos\frac{6\pi}{5}(1 - a^2\frac{3+\sqrt{5}}{2}) + a^2\frac{3+\sqrt{5}}{2} & -\frac{a}{2}\sin\frac{6\pi}{5} & (\cos\frac{6\pi}{5} - 1)a^2\frac{1+\sqrt{5}}{8} \\ \frac{a}{2}\sin\frac{6\pi}{5} & \cos\frac{6\pi}{5} & a\frac{1+\sqrt{5}}{4}\sin\frac{6\pi}{5} \\ (\cos\frac{6\pi}{5} - 1)a^2\frac{1+\sqrt{5}}{8} & -a\frac{1+\sqrt{5}}{4}\sin\frac{6\pi}{5} & \cos\frac{6\pi}{5}(1 - \frac{a^2}{4}) + \frac{a^2}{4} \end{pmatrix}$$

This has trace $\frac{1-\sqrt{5}}{2}$.

One last example shows how π_4 acts on (23)(45). This element corresponds to rotation through π with axis of rotation being the y-axis. Hence

$$\pi_4 : (23)(45) \mapsto \begin{pmatrix} \cos\pi & 0 & -\sin\pi \\ 0 & 1 & 0 \\ \sin\pi & 0 & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and this has trace -1, as required. So we have found an irreducible representation with character χ_4.

3.5 Representation Corresponding to χ_5

To find a representation with character χ_5 one can take a similar approach to that in Section 3.4.1. In the notation of Section 3.4.1, we could e.g. induce a module of D_{10} with character ϕ_4 rather than one with character ϕ_3. However, a faster approach builds on the result found in Section 3.4.2.

There is an automorphism τ of A_5 sending the element (12345) to (13452), namely the conjugation by the element (12) in S_5. In A_5, these two 5-cycles are not conjugate, but in S_5 they are (any elements of the same cycle type are conjugate in S_n). τ maps into A_5 because A_5 is normal in S_5.

The automorphism τ sends 3-cycles to other 3-cycles, and elements of the conjugacy class of (12)(34) to other elements conjugate to (12)(34). Also, τ sends elements in the conjugacy class of (12345) to elements in the conjugacy class of (13452) and vice versa.

[1]In fact, every 5-cycle corresponds to a rotation with axis similar to the two calculated so far; one coordinate is 0, another $\pm\frac{1}{2}$ and a third $\pm\frac{1+\sqrt{5}}{4}$.

An automorphism composed with a representation is again a representation, so the map $\pi_4 := \pi_3 \circ \tau$ is a representation of A_5. The conjugacy classes of (123) and (12)(34) are fixed by τ, so the character of π_5 must agree with the character of π_4 on these conjugacy classes. Also, τ swaps the two remaining conjugacy classes, so the character of π_5 is the opposite of the character of π_4 on these conjugacy classes. This is exactly χ_5, so π_5 must be irreducible and we have found an irreducible representation with character χ_5.

We have hence found a representative for each of the 5 equivalence classes of irreducible representations; one for each character in the character table of A_5.

4 Appendices

4.1 Character Table of S_5

Let $G = S_5$. Then G has 7 conjugacy classes, as follows:

Representative:	e	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
Order:	1	10	20	15	30	20	24
Centralizer order:	120	12	6	8	4	6	5

There are therefore 7 irreducible characters. Two of them are 1 dimensional; the trivial character, χ_1 and the character defined by $\chi_2(g) = sn(g)$. This is a representation because it is a homomorphism from G to the group of 1×1 complex matrices. Another character is given by

$$\chi_3(g) = |\text{fix}(g)| - 1$$

where $|\text{fix}(g)|$ is the number of i fixed by g. This is a character (See [1] and also Section 3.2).

It is a known fact that a character χ is irreducible iff $<\chi, \chi>= 1$ (See course notes p. 17). In this way it can be verified that all three characters mentioned thus far are irreducible. Hence we have the first three rows of the character table:

	e	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	4	2	1	0	0	-1	-1

It is also a known fact that the product of two characters again a character. In this manner we obtain $\chi_4 = \chi_2\chi_3$. This is also irreducible because $<\ \chi_4, \chi_4 >= 1$. Another way to form new characters from old ones is to take the symmetric and antisymmetric parts of it. By [1] pp. 196-198 these are characters and are given by:

$$\chi_S(g) = \frac{1}{2}\left(\chi^2(g) + \chi(g^2)\right)$$

and

$$\chi_A(g) = \frac{1}{2}\left(\chi^2(g) - \chi(g^2)\right)$$

Therefore the symmetric and antisymmetric parts of χ_3 are respectively

	e	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_A	6	0	0	-2	0	0	1
χ_S	10	4	1	2	0	1	0

We have

$$< \chi_A, \chi_A >= 1$$

and

$$< \chi_S, \chi_S >= \frac{100}{120} + \frac{16}{12} + \frac{1}{6} + \frac{4}{80} + \frac{0}{4} + \frac{1}{6} + \frac{0}{5} = 3$$

Hence χ_A is irreducible, but χ_S is not. However, this means χ_S is a sum of irreducible characters. We can find out which of the characters χ_1 to χ_5 are contained in χ_S:

$$< \chi_S, \chi_1 >= 1$$

$$< \chi_S, \chi_2 >= 0$$

$$< \chi_S, \chi_3 >= 1$$

$$< \chi_S, \chi_4 >= 0$$

$$< \chi_S, \chi_5 >= 0$$

Therefore χ_S contains a copy of χ_1 and χ_3 but none of the other characters found so far. Since $< \chi_S, \chi_S >= 3$, χ_S must be the sum of three irreducible characters. That means $\chi_S = \chi_1 + \chi_3 + \chi_6$ where χ_6 is the irreducible character we are looking for. Hence $\chi_6 = \chi_S - \chi_3 - \chi_1$ so

	e	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_6	5	1	-1	1	-1	1	0

This can be verified to be irreducible by seeing $< \chi_6, \chi_6 >= 1$. Finally, let $\chi_7 = \chi_2\chi_6$. This is also irreducible, so is the last row of the character table. Hence the character table of S_5 is:

	e	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	4	2	1	0	0	-1	-1
χ_4	4	-2	1	0	0	1	-1
χ_5	6	0	0	-2	0	0	1
χ_6	5	1	-1	1	-1	1	0
χ_7	5	-1	-1	1	1	-1	0

4.2 Character Table of A_5

Let $G = A_5$. First note that $|A_5| = 60$ and that G has 5 conjugacy classes, with representatives and sizes as follows:

Representative:	e	(123)	(12)(34)	(12345)	(13452)
Order:	1	20	15	12	12
Centralizer order:	60	3	4	5	5

Hence there are 5 irreducible characters. One of them is obviously the trivial representation, with the trivial character $\chi_1(g) = 1 \forall g \in G$. In order to find the other characters we must look at the characters of S_5 and restrict them to A_5 in the hopes that some of them will be irreducible. Restricting all irreducible characters of S_5 to A_5 we get:

	e	(123)	(12)(34)	(12345)	(13452)
χ_1	1	1	1	1	1
χ_2	1	1	1	1	1
χ_3	4	1	0	-1	-1
χ_4	4	1	0	-1	-1
χ_5	6	0	-2	1	1
χ_6	5	-1	1	0	0
χ_7	5	-1	1	0	0

Then we have $< \chi_3, \chi_3 >= 1$, $< \chi_5, \chi_5 >= 2$ and $< \chi_6, \chi_6 >= 1$. So we have found two irreducible characters, apart from the trivial one. We know that there are 5 irreducible characters, since there are 5 conjugacy classes in A_5.

We also know that the sum of the squares of the degrees equal the size of the group. Therefore the degrees of the last two characters satisfy:

$$1 + 4^2 + 5^2 + f_1^2 + f_2^2 = 60$$

The only solution to this is $f_1 = f_2 = 3$. We can now use column and row orthogonality relations to determine the rest of the table. For reference, label the missing values as follows:

	e	(123)	$(12)(34)$	(12345)	(13452)
χ_1	1	1	1	1	1
χ_3	4	1	0	-1	-1
χ_6	5	-1	1	0	0
χ_a	3	a_1	a_2	a_3	a_4
χ_b	3	b_1	b_2	b_3	b_4

Now $<$ column 1, column 2 $>= 3(a_1+b_1) = 0$ and $<$ column 2, column 2 $>= 3 + a_1^2 + b_1^2 = 3$ so $a_1 = b_1 = 0$

Also $<$ column 1, column 3 $>= 6+3(a_2+b_2) = 0$ and $<$ column 3, column 3 $>= 2 + a_2^2 + b_2^2 = 4$ so $a_2 = b_2 = -1$

Also $<$ column 1, column 4 $>= -3+3(a_3+b_3) = 0$ and $<$ column 4, column 4 $>= 2 + a_3^2 + b_3^2 = 5$ so $a_3^2 - a_3 - 1 = 0$ and $b_3 = 1 - a_3$. The same equations arise for column 5. The two remaining characters must be different, so the two solutions of the quadratic occur in the two characters respectively. (The order is irrelevant.) Hence the character table for A_5 is:

	e	(123)	$(12)(34)$	(12345)	(13452)
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	5	-1	1	0	0
χ_4	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_5	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

with the characters relabelled.

4.3 Character table of A_4

The character table of A_4 is even more straight forward to find than those of A_5 and S_5. A_4 has 4 conjugacy classes with the details shown in the table below.

Representative:	e	(12)(34)	(123)	(132)
Order:	1	3	4	4
Centralizer order:	12	4	3	3

The commutator subgroup of A_4 is $A_4' = \{e, (12)(34), (13)(24), (14)(23)\}$ so $A_4/A_4' \simeq Z_3$. This is a cyclic group, so has character table

	e	$(123)A_4'$	$(132)A_4'$
ψ_1	1	1	1
ψ_2	1	λ	λ^2
ψ_3	1	λ^2	λ

Inducing these characters to characters of A_4, we get three of the 4 rows. The last row can be obtained either by using column orthogonality relations, or by noting that, just like S_5 and A_5, A_4 also has a character taking the values $\chi_2(g) = |fix(g)| - 1 \ \forall g \in A_4$. Hence the full character table of A_4 is:

	e	(12)(34)	(123)	(132)
ψ_1	1	1	1	1
ψ_2	3	-1	0	0
ψ_3	1	1	λ	λ^2
ψ_4	1	1	λ^2	λ

4.4 Character table of D_{10}

Here we find the Character table of D_{10}. This group has 4 conjugacy classes, and hence 4 irreducible characters:

Representative:	e	(12)(35)	(12345)	(13452)
Order:	1	5	2	2
Centralizer order:	10	2	5	5

The commutator subgroup of D_{10} is the set of all 'rotations', so in this case the set of 5-cycles together with the identity. Therefore D_{10}/D_{10}' has order 2, so is isomorphic to Z_2. Lifting the characters of Z_2 we get

	e	(12)(35)	(12345)	(13452)
ϕ_1	1	1	1	1
ϕ_2	1	-1	1	1

Using the fact that

$$\sum_{\phi_i \text{ irreducible}} \deg(\phi_i)^2 = |D_{10}|$$

we see that the remaining two characters must have degree 2. The rest of the table can now be found using column orthogonality relations. The complete character table of D_{10} is:

	e	$(12)(35)$	(12345)	(13452)
ϕ_1	1	1	1	1
ϕ_2	1	-1	1	1
ϕ_3	2	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
ϕ_4	2	0	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$

References

[1] Liebeck, Martin and James, Gordon; *Representations and Characters of Groups*, Cambridge University Press, Cambridge, 1993

[2] Neumann, Peter, M., Stoy, Gabrielle, A. and Thompson, Edward, C.; *Groups and Geometry*, Oxford University Press, Oxford, 1994

[3] http://en.wikipedia.org/wiki/Coordinate_rotation#Three_dimensions, accessed on 1 December, 2005