CHAPTER 6

Representations of compact groups

Throughout this chapter, G denotes a compact group.

6.1. Examples of compact groups

A standard theorem in elementary analysis says that a subset of $\mathbb{C}^m$ (m a positive integer) is compact if and only if it is closed and bounded. The group $U(n) := \{\, g \in GL_n(\mathbb{C}) \mid {}^t\bar{g}g = 1 \,\}$ is a closed and bounded subset of $M_{n\times n}(\mathbb{C}) \simeq \mathbb{C}^{n^2}$. Hence $U(n)$ is a compact group.

The real orthogonal group $O_n(\mathbb{R}) := U(n) \cap GL_n(\mathbb{R})$ is another example of a compact group.

Because of the definition of group of t.d. type, there are many such groups which are compact. Some examples of compact groups of t.d. type are

$$
\begin{aligned}
&GL_n(\mathbb{Z}_p) := \{\, g \in GL_n(\mathbb{Q}_p) \mid g \in M_{n\times n}(\mathbb{Z}_p),\ \det g \in \mathbb{Z}_p^\times \,\}\\
&Sp_{2n}(\mathbb{Z}_p) = Sp_{2n}(\mathbb{Q}_p) \cap M_{2n\times 2n}(\mathbb{Z}_p)\\
&\{\, g \in Sp_{2n}(\mathbb{Q}_p) \mid g - 1 \in M_{2n\times 2n}(p^j\mathbb{Z}_p) \,\},\ j \in \mathbb{Z},\ j > 0.
\end{aligned}
$$

6.2. Finite-dimensional representations of compact groups

If G happens to be finite, then normalized Haar measure on G assigns measure $|G|^{-1}|X|$ to each nonempty subset X of G. When discussing representations of finite groups, we often used left and right invariance of Haar measure on finite groups. Many results concerning finite-dimensional continuous representations of compact groups are proved the same way as the analogous results for finite groups, simply by replacing the normalized Haar measure on the finite group by normalized Haar measure on the compact group. Note that the continuity of a finite-dimensional representation of G implies continuity of its matrix coefficients and character, so these functions are integrable.

A finite-dimensional representation of G is said to be *continous* if all of its matrix coefficients are continuous functions. Let $\mathcal{A}(G)$ be the space spanned by all matrix coefficients of continuous finite-dimensional representations of G. This is a subspace of the space $\mathcal{C}(G)$ of continuous complex-valued functions on G, and we will see in §6.3 that $\mathcal{A}(G)$ is dense in $\mathcal{C}(G)$. If G is finite, the results of Chapter 2 tell us that $\mathcal{A}(G)$ is the set of complex-valued functions on G, so this notation is consistent with the notation used for finite groups.

Theorem. *Let (π, V) be a continuous finite-dimensional representation of G. Then π is unitary.*

Proof. Let $\langle\cdot,\cdot\rangle_1$ be any inner product on V. By continuity of π, given $v,\ w \in V$, the function $g \mapsto \langle \pi(g)v, \pi(g)w\rangle_1$ is a continuous function. Hence it is integrable. Set

$\langle v, w\rangle = \int_G \langle \pi(g)v, \pi(g)w\rangle_1\, dg$, where dg is Haar measure on G. This defines an inner product on V. It follows from G-invariance of dg that $\langle \pi(g)v, \pi(g)w\rangle = \langle v, w\rangle$. qed

As shown in Chapter 1, a finite-dimensional unitary representation is completely reducible.

Corollary. *π is completely reducible.*

Let $L^2(G)$ be the space of functions on G that are square-integrable relative to Haar measure. Then $L^2(G)$ is a Hilbert space relative to the inner product $\langle f_1, f_2\rangle_{L^2} = \int_G f_1(g)\,\overline{f_2(g)}\, dg$. Of course this inner product restricts to an inner product on the subspace $\mathcal{A}(G)$.

Theorem (Orthogonality relations for matrix coefficients). *Let (π_1, V_1) and (π_2, V_2) be irreducible continuous finite-dimensional representations of G. Let $a^i_{jk}(g)$ be the matrix entries of the matrix of $\pi_i(g)$ relative to an orthonormal basis of V_i (relative to an inner product on V_i which makes π_i unitary). Then*
(1) If $\pi_1 \not\simeq \pi_2$, then $\langle a^1_{jk}, a^2_{\ell m}\rangle_{L^2} = 0$ for all j, k, ℓ and m.
(2) $\langle a^1_{jk}, a^1_{\ell m}\rangle_{L^2} = \delta_{j\ell}\delta_{km}/n_1$, where $n_1 = \dim V_1$.

Proof. Mimic the proof of Theorem 3 of Chapter 2, except where B is a linear transformation from V_2 to V_1, set $A = \int_G \pi_1(g) B \pi_2(g)^{-1}\, dg$. Throughout the rest of the proof, replace the expression $|G|^{-1}\sum_{g\in G}$ by integration over G (relative to normalized Haar measure on G). qed

Corollary. *Let (π_1, V_1) and (π_2, V_2) be as in the theorem. If $\pi_1 \not\simeq \pi_2$, then the subspace of $\mathcal{A}(G)$ spanned by all matrix coefficients of π_1 is orthogonal to the subspace spanned by all matrix coefficients of π_2.*

The following is equivalent to part 2 of the above theorem.

Corollary. *Let (π, V) be an irreducible continuous finite-dimensional representation of G. Let v, w, v', $w' \in V$, and let $\langle\cdot,\cdot\rangle$ be a G-invariant inner product on V. Then*

$$\int_G \langle \pi(g)w, v\rangle \overline{\langle \pi(g)w', v'\rangle}\, dg = (\dim V)^{-1}\langle w, w'\rangle\langle v', v\rangle.$$

Theorem. *Let (π_1, V_1) and (π_2, V_2) be (finite-dimensional continuous) representations of G. Then*

$$\int_G \chi_{\pi_1}(g)\overline{\chi_{\pi_2}(g)}\, dg = \dim \mathrm{Hom}(\pi_1, \pi_2).$$

If π_1 and π_2 are irreducible, then

$$\int_G \chi_{\pi_1}(g)\overline{\chi_{\pi_2}(g)}\, dg = \begin{cases} 1, & \text{if } \pi_1 \simeq \pi_2 \\ 0, & \text{otherwise.} \end{cases}.$$

6.3. The Peter-Weyl Theorem

As above, let $L^2(G)$ be the space of measurable functions on G for which $\int_G |f(g)|^2\,dg < \infty$. (Here, the integral is relative to Haar measure on G). If $f \in L^2(G)$, set $\|f\|_2 = \left(\int_G |f(g)|^2\,dg\right)^{1/2}$. Let $L^1(G)$ be the space of measurable functions on G for which $\int_G |f(g)|\,dg < \infty$. For $f \in L^1(G)$, set $\|f\|_1 = \int_G |f(g)|\,dg$. If $f_1,\ f_2 \in L^2(G)$, then $f_1 f_2 \in L^1(G)$ and $\|f_1 f_2\|_1 \le \|f_1\|_2 \|f_2\|_2$. Given $f_1,\ f_2 \in L^2(G)$, the inner product $\langle f_1, f_2\rangle_{L^2}$ is defined by

$$\langle f_1, f_2\rangle_{L^2} = \int_G f_1(g)\,\overline{f_2(g)}\,dg.$$

With this inner product, $L^2(G)$ is a Hilbert space. The inequality $|\langle f_1, f_2\rangle_{L^2}| \le \|f_1\|_2 \|f_2\|_2$ is known as the Schwarz inequality.

If $f : G \to \mathbb{C}$ and $g \in G$, set $(\mathcal{L}(g)f)(g_0) = f(g^{-1}g_0)$, $g_0 \in G$.

Lemma 1. *Let $f \in L^2(G)$. Then the map $g \mapsto \mathcal{L}(g)f$ is a continuous map from G to $L^2(G)$.*

Proof. It is clear that the function $\mathcal{L}(g)f$ belongs to $L^2(G)$. Let $\varepsilon > 0$. Using the fact that $\mathcal{C}(G)$ is dense in $L^2(G)$, choose a continuous function $\varphi \in \mathcal{C}(G)$ such that $\|f - \varphi\|_2 < \varepsilon/3$. Because G is compact and φ is continuous, φ is uniformly continuous. Hence there exists an open neighbourhood U of 1 in G such that if $g_1^{-1}g_2 \in U$, then $|\varphi(g_1^{-1}g) - \varphi(g_2^{-1}g)| < \varepsilon/3$ for all $g \in G$. Now

$$\begin{aligned}\|\mathcal{L}(g_1)f - \mathcal{L}(g_2)f\|_2 &\le \|\mathcal{L}(g_1)f - \mathcal{L}(g_1)\varphi\|_2 + \|\mathcal{L}(g_1)\varphi - \mathcal{L}(g_2)\varphi\|_2 + \|\mathcal{L}(g_2)\varphi - \mathcal{L}(g_2)f\|_2\\ &\le 2\|\varphi - f\|_2 + \|\mathcal{L}(g_1)\varphi - \mathcal{L}(g_2)\varphi\|_2\\ &< 2\varepsilon/3 + \sup_{g\in G}|\mathcal{L}(g_1)\varphi - \mathcal{L}(g_2)\varphi| < \varepsilon\end{aligned}$$

Lemma 2. *Let $f \in L^2(G)$. For every $\varepsilon > 0$, there exist finitely many $g_i \in G$ and Borel sets $S_i \subset G$ such that G is the disjoint union of the S_i's and $\|\mathcal{L}(g)f - \mathcal{L}(g_i)f\|_2 < \varepsilon$ for all i and for all $g \in S_i$.*

Proof. By Lemma 1, there exists an open neighbourhood U of 1 such that $\|\mathcal{L}(g)f - f\|_2 < \varepsilon$ for all $g \in U$. Let $g \in U$ and $g_0 \in G$. Then $\|\mathcal{L}(gg_0)f - \mathcal{L}(g_0)f\|_2 < \varepsilon$. Now Ug_0 is an open neighbourhood of g_0. Hence $\{\,Ug_0 \mid g_0 \in G\,\}$ is an open cover of G. By compactness of G, there are finitely many $g_1, \dots, g_n$ such that $G = \cup_{i=1}^n Ug_i$. Let $S_j = \left(U_j \setminus \cup_{i=1}^{j-1} U_i\right)^{-1}$. qed

Lemma 3. *Let $f \in L^2(G)$ and $\dot{f} \in L^1(G)$. Set $F(g_0) = \int_G \dot{f}(g) f(g^{-1}g_0)\,dg$. Then F is a limit in $L^2(G)$ of a sequence of functions, each of which is a finite linear combination of left translates of f.*

Proof. Let $\varepsilon > 0$. Choose g_i and S_i as in Lemma 2. Set $c_i = \int_{S_i} \dot{f}(g)\, dg$. Then

$$\begin{aligned} \|F - \sum_{i=1}^{n} c_i \mathcal{L}(g_i) f\|_2 &\le \sum_{i=1}^{n} \int_{S_i} |\dot{f}(g)| \|\mathcal{L}(g)f - \mathcal{L}(g_i)f\|_2 \, dg \\ &\le \sum_{i=1}^{n} \int_{S_i} |\dot{f}(g)| \varepsilon \, dg = \varepsilon \|f\|_1. \end{aligned}$$

qed

As in §6.2, the space spanned by all matrix coefficients of continuous finite-dimensional representations of G is denoted by $\mathcal{A}(G)$.

Theorem (Peter-Weyl Theorem). *$\mathcal{A}(G)$ is dense in $L^2(G)$.*

Proof. Let (π, V) be a continuous finite-dimensional representation that is unitary relative to an inner product $\langle \cdot, \cdot \rangle$ on V. Then, given v, $w \in V$, the function $g \mapsto \langle \pi(g)v, w \rangle$ is a matrix coefficient of π. Let $f(g) = \langle \pi(g)v, w \rangle$, $g \in G$. The functions $g \mapsto \overline{f(g^{-1})}$, $g \mapsto f(gg_0)$ and $g \mapsto f(g_0 g)$ are also matrix coefficients of π. Hence the space $\mathcal{A}(G)$ is closed under the map taking f to the function $g \mapsto \overline{f(g^{-1})}$ and also under left and right translation. The closure $\overline{\mathcal{A}(G)}$ of $\mathcal{A}(G)$ in $L^2(G)$ is also invariant under these operations.

Suppose that $\overline{\mathcal{A}(G)} \neq L^2(G)$. Then $\overline{\mathcal{A}(G)}^{\perp} \neq \{0\}$, and $\overline{\mathcal{A}(G)}^{\perp}$ is closed under the above operations. Suppose that $f_0 \in \overline{\mathcal{A}(G)}$ is nonzero. If U is an open neighbourhood of 1, let 1_U be the characteristic function of U, and let $|U|$ be the Haar measure of U. Define $f_U(g) = |U|^{-1} \int_G 1_U(g_0) f_0(g_0^{-1} g)\, dg_0$. Since 1_U, $f_0 \in L^2(G)$, the Schwarz inequality can be used to prove that f_U is a continuous function. As U shrinks to $\{1\}$, $f_U \to f_0$ in $L^2(G)$. Because $f_0 \neq 0$, we know that some f_U is nonzero. Since $\overline{\mathcal{A}(G)}$ is G-invariant (that is, invariant under left translation by elements of G) and the left regular representation is unitary, it follows that $\overline{\mathcal{A}(G)}^{\perp}$ is also G-invariant. Hence linear combinations of translates of f_0 belong to $\overline{\mathcal{A}(G)}^{\perp}$. Applying Lemma 3, we see that $f_U \in \overline{\mathcal{A}(G)}^{\perp}$. Therefore $\overline{\mathcal{A}(G)}^{\perp}$ contains a nonzero continuous function.

After translating and multiplying by a scalar, if necessary, we can produce a continuous function $f_1 \in \overline{\mathcal{A}(G)}^{\perp}$ such that $f_1(1)$ is a nonzero real number. Set $f_2(g) = \int_G f_1(g_0 g g_0^{-1})\, dg_0$. Then f_2 is continuous, f_2 is a class function, and $f_2(1)$ is a nonzero real number. To see that $f_2 \in \overline{\mathcal{A}(G)}^{\perp}$, note that if $h \in \overline{\mathcal{A}(G)}$ then

$$\begin{aligned} \langle f_2, h \rangle_{L^2} = \int_G f_2(g)\, \overline{h(g)}\, dg &= \int_G \int_G f_1(g_0 g g_0^{-1})\, \overline{h(g)}\, dg_0\, dg \\ &= \int_G \int_G f_1(g) \overline{h(g_0^{-1} g g_0)}\, dg\, dg_0 = 0 \end{aligned}$$

because the function $g \mapsto h(g_0^{-1} g g_0)$ belongs to $\overline{\mathcal{A}(G)}$.

Next, set $f_3(g) = f_2(g) + \overline{f_2(g^{-1})}$. Then $f_3(g) = \overline{f_3(g^{-1})}$. Let $K(g, g_0) = f_3(g^{-1}g_0)$, $g,$, $g_0 \in G$. Set

$$(Tf)(g) = \int_G K(g, g_0)f(g_0)\,dg_0, \qquad f \in L^2(G).$$

Then T is an operator on $L^2(G)$. Note that $\overline{K(g, g_0)} = \overline{f_3(g^{-1}g_0)} = f_3(g_0^{-1}g) = K(g_0, g)$. Thus T is a Hilbert-Schmidt operator(see comments following the proof). Now T is nonzero, so, according to the Hilbert-Schmidt Theorem, T has a nonzero real eigenvalue λ. and the eigenspace $V_\lambda \subset L^2(G)$ is finite-dimensional. If $f \in V_\lambda$, then

$$\begin{aligned}(T\mathcal{L}(g_0)f)(g) &= \int_G K(g, g_1)(\mathcal{L}(g_0)f)(g_1)\,dg_1 = \int_G f_3(g^{-1}g_1)f(g_0^{-1}g_1)\,dg_1 \\ &= \int_G f_3(g^{-1}g_0g_1)f(g_1)\,dg_1 = \int_G K(g_0^{-1}g, g_1)f(g_1)\,dg_1 \\ &= (Tf)(g_0^{-1}g) = \lambda\, f(g_0^{-1}g) = \lambda\,(\mathcal{L}(g_0)f)(g).\end{aligned}$$

Hence $T\mathcal{L}(g_0)f = \lambda\mathcal{L}(g_0)f$ for all $f \in V_\lambda$. Therefore V_λ is invariant under left translation. According to Lemma 1, $g \mapsto \mathcal{L}(g)f$ is continuous. Because the restriction of left translation to V_λ is a finite-dimensional representation of G, there exists a finite-dimensional irreducible G-invariant (nonzero) subspace W of V_λ. Choose an orthonormal basis $\{\, e_1, \ldots, e_n \,\}$ of W (with respect to which left translation is unitary). Then $g \mapsto \langle \mathcal{L}(g)e_j, e_i\rangle_{L^2} = \int_G e_j(g^{-1}g_0)\overline{e_i(g_0)}\,dg_0$ is a matrix coefficient of this representation, hence belongs to $\mathcal{A}(G)$. Since $f_3 \in \overline{\mathcal{A}(G)}^{\perp}$, we have

$$\begin{aligned}0 &= \int_G f_3(g)\left(\int_G \overline{e_j(g^{-1}g_0)}e_j(g_0)\,dg_0\right)dg = \int_G \left(\int_G f_3(g)\overline{e_j(g^{-1}g_0)}\,dg\right)e_j(g_0)\,dg_0 \\ &= \int_G \int_G \left(f_3(g_0g^{-1})\overline{e_j(g)}\,dg\right)e_j(g_0)\,dg_0 = \int_G \left(\int_G f_3(g^{-1}g_0)e_j(g_0)dg_0\right)\overline{e_j(g)}\,dg \\ &= \int_G (Te_j)(g)\overline{e_j(g)}\,dg = \lambda\langle e_j, e_j\rangle_{L^2}\end{aligned}$$

(For the fourth equality above, we used the fact that f_3 is a class function: $f_3(g_0g^{-1}) = f_3(g^{-1}(g_0g^{-1})g) = f_3(g^{-1}g_0)$). Hence $\lambda = 0$. Contradiction. Hence the assumption that $\overline{\mathcal{A}(G)}^{\perp}$ is nonzero is false. qed

If $K \in L^2(G\times G)$, then the operator on $L^2(G)$ defined by $(Tf)(g) = \int_G K(g, g_0)f(g_0)\,dg_0$ is called a *Hilbert-Schmidt* operator. Such an operator is self-adjoint when $\overline{K(g, g_0)} = K(g_0, g)$ for all g, $g_0 \in G$. The Hilbert-Schmidt Theorem is a theorem from functional analysis which says that a nonzero self-adjoint Hilbert-Schmidt operator has some nonzero eigenvalues, and the eigenspace corresponding to each nonzero eigenvalue is finite-dimensional. For a proof of the Hilbert-Schmidt Theorem, see Appendix B of the book by Simon (listed in the references).

As we will soon see, the Peter-Weyl Theorem can be applied to prove that an infinite-dimensional unitary representation of a compact group is a direct sum of finite-dimensional irreducible representations. Let $\widehat{G}$ be a collection of finite-dimensional irreducible unitary representations of G having the property that each finite-dimensional irreducible representation of G is equivalent to exactly one of the representations in $\widehat{G}$. If $\rho \in \widehat{G}$, let d_ρ be the degree of ρ, and let $a_\rho(g)_{ij}$ be the matrix coefficients of ρ relative to an orthonormal basis of the space of ρ, such that the corresponding inner product on the space of ρ is G-invariant.

Let $\mathcal{H}$ be a separable Hilbert space. Let π be a unitary representation of G in the space $\mathcal{H}$, having the property that the map $(g, v) \mapsto \pi(g)v$ is a continuous map from $G \times \mathcal{H}$ to $\mathcal{H}$. Define a linear operator P^ρ_{ij} on $\mathcal{H}$ by $P^\rho_{ij}(v) = d_\alpha^{-1} \int_G \overline{a^\rho(g)_{ij}}\, \pi(g)v\, dg$, $v \in \mathcal{H}$. This is the analogue of the definition used in Chapter 2 when studying the decompositions of representations of finite groups. Arguing as in Chapter 2, we can easily prove:

Lemma 4. *Let $\rho \in \widehat{G}$. Let π be as above.*
(1) If $g \in G$, then $\pi(g)P^\rho_{ij} = \sum_k a^\rho(g)_{ki} P^\rho_{kj}$
(2) $P^\rho_{mn} \circ P^\rho_{ij} = \delta^\rho_{ni} P^\rho_{mj}$. If $\rho' \in \widehat{G}$ and $\rho' \neq \rho$, then $P^{\rho'}_{mn} \circ P^\rho_{ij} = 0$ for all m, n, i, and j.
(3) $(P^\rho_{ij})^* = P^\rho_{ji}$

Let $P^\rho = \sum_i P^\rho_{ii} = d_\rho^{-1} \int_G \overline{\chi_\rho(g)}\pi(g)\, dg$. (Here, χ_ρ denotes the character of ρ.)

Lemma 5.
(1) $P^\rho \in \mathrm{Hom}_G(\pi, \pi)$,
(2) If $\rho' \in \widehat{G}$ and $\rho' \neq \rho$, then $P^\rho \circ P^{\rho'} = 0$.
(3) $(P^\rho)^2 = P^\rho$ and $(P^\rho)^* = P^\rho$.

It follows from Lemma 5 that $\{ P^\rho \mid \rho \in \widehat{G} \}$ is a set of mutually orthogonal projections.

Theorem. *Let $\mathcal{H}$ be a separable Hilbert space. Let π be a unitary representation of G in the space $\mathcal{H}$, having the property that the map $(g, v) \mapsto \pi(g)v$ is a continuous map from $G \times \mathcal{H}$ to $\mathcal{H}$. Then $\mathcal{H} = \oplus_{n=1}^\infty \mathcal{H}_n$, where each $\mathcal{H}_n$ is a finite-dimensional G-invariant irreducible subspace of $\mathcal{H}$.*

Proof. Let $\langle \cdot, \cdot \rangle$ be a G-invariant inner product on $\mathcal{H}$. Suppose that $v \in \mathcal{H}$ belongs to $P^\rho(\mathcal{H})^\perp$ for all $\rho \in \widehat{G}$. Because $\langle P^\rho(v), v \rangle = 0$ and $I - P^\rho$ is the orthogonal projection of $\mathcal{H}$ onto $P^\rho(\mathcal{H})^\perp$, we have $(I - P^\rho)(v) = v$. From Lemma 4(2), we can see that $P^\rho_{mn}(I - P^\rho) = 0$. It follows that $P^\rho_{mn}(v) = 0$ for all $\rho \in \widehat{G}$ and all m and n.

Let $\{ e_k \}$ be an orthonormal basis of $\mathcal{H}$. Set $f_k(g) = \langle e_k, \pi(g)v \rangle$. Our hypotheses guarantee that f_k is a continuous function, hence lies in $L^2(G)$. By the Peter-Weyl theorem, we have an L^2-expansion

$$f_k(g) = \sum_{\rho; ij} c_{\rho;ij}(k) a^\rho(g)_{ij},$$

with $c_{\rho;ij}(k) = \langle e_k, P^\rho_{ij}(v)\rangle$. Because $P^\rho_{ij}(v) = 0$ for all ρ, i and j, it follows that $c_{\rho;ij}(k) = 0$ for all k. Therefore $f_k(g) = 0$ for almost all g. Continuity of f_k then guarantees that f_k is identically zero for all k. In particular, $f_k(1) = \langle e_k, v\rangle = 0$ for all k. Hence $v = 0$.

We have shown that the only vector with is orthogonal to every $P^\rho(\mathcal{H})$ is the zero vector. Hence $\mathcal{H}$ is the direct sum of the subspaces $P^\rho(\mathcal{H})$. According to Lemma 5(1), $P^\rho(\mathcal{H})$ is a G-invariant subspace of $\mathcal{H}$. Hence to complete the proof of the theorem it suffices to prove that the restriction π_ρ of π to $P^\rho(\mathcal{H})$ is a direct sum of irreducible representations of G, for every $\rho \in \widehat{G}$. Let $m_\rho = \dim P^\rho_{11}(\mathcal{H})$. Arguing as in Chapter 2, we can show that $\dim P^\rho_{ii}(\mathcal{H}) = m_\rho$, for all i, and π_ρ is equivalent to the representation $m_\rho\rho$ (the m_ρ-fold direct sum $\rho \oplus \cdots \oplus \rho$ -m_ρ times). qed

Corollary. *A continuous irreducible unitary representation of G is finite-dimensional.*

6.4. Weyl's character formula

Let G be a compact connected Lie group. A *Cartan subgroup* of such a group G is a closed subgroup of G that is isomorphic to a direct product of the form $\mathbb{R}/\mathbb{Z} \times \cdots \times \mathbb{R}/\mathbb{Z}$ and is maximal with respect to this property. (That is, it cannot be a proper subgroup of another closed subgroup that is also isomorphic to a direct product of the same form). A Cartan subgroup is a maximal abelian subgroup and consists of semisimple elements. (Recall that an element of a matrix Lie group is semisimple if it is semisimple as a matrix, that is, it is diagonalizable over the complex numbers.) A Cartan subgroup is equal to its own centralizer in G. The Cartan subgroups of G are all conjugate in G.

Examples:

(1) $G = U(n)$. The subgroup T of diagonal matrices in G consists of diagonal matrices in $GL_n(\mathbb{C})$ whose diagonal entries are $e^{i\theta_1}, \ldots, e^{i\theta_n}$ for some $\theta_1, \ldots, \theta_n \in \mathbb{R}$. The group T is a Cartan subgroup of $U(n)$.

(2) Let $G = SO_4(\mathbb{R}) = O_4(\mathbb{R}) \cap SL_4(\mathbb{R})$. The subgroup

$$T = \left\{ \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & 0 & \sin\theta_2 & \cos\theta_2 \end{pmatrix} \;\middle|\; \theta_1,\, \theta_2 \in \mathbb{R} \right\}$$

is a Cartan subgroup of G

Fix a Cartan subgroup T of G. The set $\hat{T}$ of unitary characters of T is a locally compact group with the discrete topology. Let $N_G(T)$ be the normalizer of T in G, and $W(G,T) = N_G(T)/T$. The group $W(G,T)$ is a finite group, called the *Weyl group of* T. Each $w \in W(G,T)$ acts on T and hence on $\hat{T}$ by $(w \cdot \chi)(t) = \chi(w^{-1} \cdot t)$, $\chi \in \hat{T}$, $t \in T$. Let $\hat{T}'$ be the set of $\chi \in \hat{T}$ such that $w \cdot \chi \neq \chi$ for any nontrivial $w \in W$.

The set of irreducible unitary representations of G can be parametrized by the set of $W(G,T)$-orbits in $\hat{T}'$. Given $\chi \in \hat{T}'$, there is an irreducible unitary representation π_χ of G. Every irreducible unitary representation of G is equivalent to π_χ for some $\chi \in \hat{T}'$. Furthermore, given χ, $\dot{\chi} \in \hat{T}'$, π_χ and $\pi_{\dot{\chi}}$ are equivalent if and only if $\dot{\chi} = w \cdot \chi$ for some $w \in W(G,T)$.

Weyl's character formula. *Let* $\chi \in \hat{T}'$. *Let* Θ_χ *be the character of* π_χ. *The function* Θ_χ *is determined by its values on* T. *There is a sign* $\epsilon(\chi) = \pm 1$ *such that*

$$\begin{aligned}\Theta_\chi(t) &= \epsilon(\chi) \sum_{w \in W(G,T)} \Delta_G(w \cdot t)^{-1} \chi(w \cdot t), \\ &= \epsilon(\chi) \Delta_G(t)^{-1} \sum_{w \in W(G,T)} \det(w)(w^{-1} \cdot \chi)(t), \qquad t \in T'.\end{aligned}$$

Here, $T' = \{\, t \in T \mid w \in W(G,T),\ w \neq 1 \Rightarrow w \cdot t \neq t, \,\}$ *and* Δ_G *is the so-called Weyl denominator. It has the property that* $\Delta_G(w \cdot t) = \det(w)\Delta(t)$, $t \in T'$, *with* $\det(w) = \pm 1$.