CHAPTER 5

Topological Groups, Representations, and Haar Measure

5.1. Topological spaces

If X is a set, a family \mathcal{U} of subsets of X defines a *topology* on X if

- (i) $\emptyset \in \mathcal{U}, X \in \mathcal{U}.$
- (ii) The union of any family of sets in \mathcal{U} belongs to \mathcal{U} .
- (iii) THe intersection of a finite number of sets in \mathcal{U} belongs to \mathcal{U} .

If \mathcal{U} defines a topology on X, we say that X is a *topological space*. The sets in \mathcal{U} are called *open sets*. The sets of the form $X \setminus U$, $U \in \mathcal{U}$, are called *closed sets*. If Y is a subset of X the *closure* of Y is the smallest closet set in X that contains Y.

Let Y be a subset of a topological space X. Then we may define a topology \mathcal{U}_Y on Y, called the *subspace* or *relative* topology, or the topology on Y *induced* by the topology on X, by taking $\mathcal{U}_Y = \{Y \cap U \mid U \in \mathcal{U}\}.$

A system \mathcal{B} of subsets of X is called a *basis* (or *base*) for the topology \mathcal{U} if every open set is the union of certain sets in \mathcal{B} . Equivalently, for each open set U, given any point $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

Example: The set of all bounded open intervals in the real line \mathbb{R} forms a basis for the usual topology on \mathbb{R} .

Let $x \in X$. A neighbourhood of x is an open set containing x. Let \mathcal{U}_x be the set of all neighbourhoods of x. A subfamily \mathcal{B}_x of \mathcal{U}_x is a basis or base at x, a neighbourhood basis at x, or a fundamental system of neighbourhoods of x, if for each $U \in \mathcal{U}_x$, there exists $B \in \mathcal{B}_x$ such that $B \subset U$. A topology on X may be specified by giving a neighbourhood basis at every $x \in X$.

If X and Y are topological spaces, there is a natural topology on the Cartesian product $X \times Y$ that is defined in terms of the topologies on X and Y, called the *product topology*. Let $x \in X$ and $y \in Y$. The sets $U_x \times V_y$, as U_x ranges over all neighbourhoods of x, and V_y ranges over all neighbourhoods of y forms a neighbourhood basis at the point $(x, y) \in X \times Y$ (for the product topology).

If X and Y are topological spaces, a function $f: X \to Y$ is *continuous* if whenever U is an open set in Y, the set $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$ is an open set in X. A function $f: X \to Y$ is a *homeomorphism* (of X onto Y) if f is bijective and both f and f^{-1} are continuous functions.

An open covering of a topological space X is a family of open sets having the property that every $x \in X$ is contained in at least one set in the family. A *subcover* of an open covering is a an open covering of X which consists of sets belonging to the open covering. A topological space X is *compact* if every open covering of X contains a finite subcover. A subset Y of a topological space X is *compact* if it is compact if Y is compact in the subspace topology. A topological space X is *locally compact* if for each $x \in X$ there exists a neighbourhood of x whose closure is compact.

A topological space X is Hausdorff (or T_2) if given distinct points x and $y \in X$, there exist neighbourhoods U of x and V of y such that $U \cap V = \emptyset$. A closed subset of a locally compact Hausdorff space is locally compact.

5.2. Topological groups

A topological group G is a group that is also a topological space, having the property the maps $(g_1, g_2) \mapsto g_1 g_2$ from $G \times G \to G$ and $g \mapsto g^{-1}$ from G to G are continuous maps. In this definition, $G \times G$ has the product topology.

Lemma. Let G be a topological group. Then

- (1) The map $g \mapsto g^{-1}$ is a homeomorphism of G onto itself.
- (2) Fix $g_0 \in G$. The maps $g \mapsto g_0 g$, $g \mapsto g g_0$, and $g \mapsto g_0 g g_0^{-1}$ are homeomorphisms of G onto itself.

A subgroup H of a topological group G is a topological group in the subspace topology. Let H be a subgroup of a topological group G, and let $p: G \to G/H$ be the canonical mapping of G onto G/H. We define a topology $\mathcal{U}_{G/H}$ on G/H, called the *quotient topology*, by $\mathcal{U}_{G/H} = \{ p(U) \mid U \in \mathcal{U}_G \}$. (Here, \mathcal{U}_G is the topology on G). The canonical map p is open (by definition) and continuous. If H is a closed subgroup of G, then the topological space G/H is Hausdorff. If H is a normal subgroup of G, then G/H is a topological group.

If G and G' are topological groups, a map $f: G \to G'$ is a continuous homomorphism of G into G' if f is a homomorphism of groups and f is a continuous function. If H is a closed normal subgroup of a topological group G, then the canonical mapping of G onto G/H is an open continuous homomorphism of G onto G/H.

A topological group G is a *locally compact group* if G is locally compact as a topological space.

Proposition. Let G be a locally compact group and let H be a closed subgroup of G. Then

- (1) H is a locally compact group (in the subspace topology).
- (2) If H is normal in G, then G/H is a locally compact group.
- (3) If G' is a locally compact group, then $G \times G'$ is a locally compact group (in the product topology).

5.3. General linear groups and matrix groups

Let F be a field that is a topological group (relative to addition). Assume that points in F are closed sets in the topology on F. For example, we could take $F = \mathbb{R}$,

 \mathbb{C} or the *p*-adic numbers \mathbb{Q}_p , *p* prime. Let *n* be a positive integer. The space $M_{n\times n}(F)$ of $n \times n$ matrices with entries in *F* is a topological group relative to addition, when $M_{n\times n}(F) \simeq F^{n^2}$ is given the product topology. The multiplicative group $GL_n(F)$, being a subset (though not a subgroup) of $M_{n\times n}(F)$, is a topological space in the subspace topology. The determinant map det from $M_{n\times n}(F)$ to *F*, being a polynomial in matrix entries, is a continuous function. Now $F^{\times} = F \setminus \{0\}$ is an open subset of *F* (since points are closed in *F*). Therefore, by continuity of det, $GL_n(F) = \det^{-1}(F^{\times})$ is an open subset of $M_{n\times n}(F)$. It is easy to show that matrix multiplication, as a map (not a homomorphism) from $M_{n\times n}(F) \times M_{n\times n}(F)$ to $M_{n\times n}(F)$ is continuous. It follows that the restriction to $GL_n(F) \times GL_n(F)$ is also continuous. Let $g \in GL_n(F)$. Recall that Cramer's rule gives a formula for the *ij*th entry of g^{-1} as the determinant of the matrix given by deleting the *i*th row and *j*th column of *g*, divided by det *g*. Using this, we can prove that $g \mapsto g^{-1}$ is a continuous map from $GL_n(F)$ to $GL_n(F)$. Therefore $GL_n(F)$ is a topological group. We can also see that if *F* is a locally compact group (for example if $F = \mathbb{R}$, \mathbb{C} or \mathbb{Q}_p), then $GL_n(F)$ is a locally compact group.

The group $SL_n(F)$, being the kernel of the continuous homomorphism det : $GL_n(F) \to F^{\times}$, is a closed subgroup of $SL_n(F)$, so is a locally compact group whenever F is a locally compact group. If I_n is the $n \times n$ identity matrix and $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, then $Sp_{2n}(F) = \{g \in GL_{2n}(F) \mid {}^tgJg = J\}$ is a closed subgroup of $GL_{2n}(F)$. If $S \in GL_n(F)$ is a symmetric matrix, that is ${}^tS = S$, the group $O_n(S) = \{g \in GL_n(F) \mid {}^tgSg = S\}$ is an orthogonal group, and is a closed subgroup of $GL_n(F)$. Depending on the field F, different choices of S can give rise to non-isomorphic orthogonal groups. If E is a quadratic extension of F and $X \in M_{n \times n}(E)$, let \overline{X} be the matrix obtained from X by letting the nontrivial element of the Galois group Gal(E/F) act on each the entries of X. Suppose that $h \in GL_n(E)$ is a matrix such that ${}^t\overline{h} = h$ (h is hermitian). Then the group $U(h) = \{g \in GL_{2n}(E) \mid {}^t\overline{g}hg = h\}$ is called a unitary group and is a closed subgroup of $GL_n(E)$. If (n_1, \ldots, n_r) is a partition of n then the corresponding standard parabolic subgroup $P = P_{(n_1, \ldots, n_r)}$ of $GL_n(F)$ is a closed subgroup of $GL_n(F)$, as are any Levi factor of P, and the unipotent radical of P.

5.4. Matrix Lie groups

A Lie group is a topological group that is a differentiable manifold with a group structure in which the multiplication and inversion maps from $G \times G$ to G and from G to Gare smooth maps. Without referring to the differentiable manifolds, we may define a matrix Lie group, or a closed Lie subgroup of $GL_n(\mathbb{C})$ to be a closed subgroup of the topological group $GL_n(\mathbb{C})$. (This latter definition is reasonable because $GL_n(\mathbb{C})$ is a Lie group, and it can be shown that a closed subgroup of a Lie group is also a Lie group). A connected matrix Lie group is reductive if it is stable under conjugate transpose, and semisimple if it is reductive and has finite centre. The book of Hall [Hall] gives an introduction to matrix Lie groups, their structure, and their finite-dimensional representations. For other references on Lie groups and their representations, see [B], [K1] and [K2].

5.5. Finite-dimensional representations of topological groups and matrix Lie groups

Let G be a topological group. A (complex) finite-dimensional *continuous* representation of G is a finite-dimensional (complex) representation (π, V) of G having the property that the map $g \mapsto [\pi(g)]_{\beta}$ from G to $GL_n(\mathbb{C})$ is a continuous homomorphism for some (hence any) basis β of V. The continuity property is equivalent to saying that every matrix coefficient of π is a continuous function from G to \mathbb{C} . Hence to prove the following lemma, we need only observe that the character of π is a finite sum of matrix coefficients of π .

Lemma. Let π be a continuous finite-dimensional representation of G. Then the character χ_{π} of π is a continuous function on G.

Example: Let π be a continuous one-dimensional representation of the locally compact group \mathbb{R} . Then π is a continuous function from \mathbb{R} to \mathbb{C} such that $\pi(0) = 1$ and $\pi(t_1 + t_2) = \pi(t_1)\pi(t_2)$ for all $t_1, t_2 \in \mathbb{R}$. If $f : \mathbb{R} \to \mathbb{C}$ is continuously differentiable and the support of f is contained in a compact subset of \mathbb{R} , then $\int_{-\infty}^{\infty} f(t)\pi(t) dt$ converges. Choose f so that $c = \int_{-\infty}^{\infty} f(t)\pi(t) dt \neq 0$. Multiplying $\pi(t_1 + t_2)$ by $f(t_2)$ and integrating, we have

$$\int_{-\infty}^{\infty} f(t_2)\pi(t_1+t_2) \, dt_2 = \pi(t_1) \int_{-\infty}^{\infty} f(t_2)\pi(t_2) \, dt_2 = c\pi(t_1), \qquad t_1 \in \mathbb{R}.$$

Then

$$\pi(t_1) = c^{-1} \int_{-\infty}^{\infty} f(t_2) \pi(t_1 + t_2) \, dt_2 = c^{-1} \int_{-\infty}^{\infty} \pi(t) f(t - t_1) \, dt, \qquad t_1 \in \mathbb{R}$$

Because $t_1 \mapsto \int_{-\infty}^{\infty} \pi(t) f(t-t_1) dt$ is a differentiable function of t_1 , we see that π is a differentiable function. Differentiating both sides of $\pi(t_1 + t_2) = \pi(t_1)\pi(t_2)$ with respect to t_1 and then setting $t_1 = 0$ and $t = t_2$, we obtain $\pi'(t) = \pi'(0)\pi(t)$. Setting $k = \pi'(0)$, we have $\pi'(t) = k\pi(t), t \in \mathbb{R}$. Solving this differential equation yields $\pi(t) = ae^{kt}$ for some $a \in \mathbb{C}$. And $\pi(0) = 1$ forces a = 1. Hence $\pi(t) = e^{kt}$. Now if we take a $z \in \mathbb{C}$, it is clear that $t \mapsto e^{zt}$ is a one-dimensional continuous representation of \mathbb{R} .

Lemma. Let $z \in \mathbb{C}$. Then $\pi_z(t) = e^{zt}$ defines a one-dimensional continuous representation of \mathbb{R} . The representation π_z is unitary if and only if the real part of z equals 0. Each one-dimensional continuous representation of \mathbb{R} is of the form π_z for some $z \in \mathbb{C}$, and any one-dimensional continuous representation of \mathbb{R} is a smooth (infinitely differentiable) function of t.

Theorem. A continuous homomorphism from a Lie group G to a Lie group G' is smooth.

Let G be a Lie group (for example, a matrix Lie group). Then, because $GL_n(\mathbb{C})$ is a Lie group, via a choice of basis for the space of the representation, a finite-dimensional representation of G is a continuous homomorphism from G to $GL_n(\mathbb{C})$. According to the theorem, the representation must be a smooth map from G to $GL_n(\mathbb{C})$.

Corollary. A continuous finite-dimensional representation of a Lie group is smooth.

Definition. A Lie algebra over a field F is a vector space \mathfrak{g} over F endowed with a bilinear map, the Lie bracket, denoted $(X, Y) \mapsto [X, Y] \in \mathfrak{g}$ satisfying [X, Y] = -[Y, X] and the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \qquad \forall X, Y, Z \in \mathfrak{g}.$$

If G is a Lie group, the Lie algebra \mathfrak{g} of G is defined to be the set of left-G-invariant smooth vector fields on G. A vector field is a smoothly varying family of tangent vectors, one for each $g \in G$, and it can be shown that if X is identified with the corresponding tangent vector at the identity element, then the Lie algebra \mathfrak{g} is identified with the tangent space at the identity.

If we work with matrix Lie groups, we can take a different approach. If $X \in M_{n \times n}(\mathbb{C})$, then the matrix exponential $e^X = \sum_{k=0}^{\infty} X^k / k!$ is an element of $GL_n(\mathbb{C})$.

Proposition. Let G be a matrix Lie group. Then the Lie algebra \mathfrak{g} is equal to

$$\mathfrak{g} = \{ X \in M_{n \times n}(\mathbb{C}) \mid e^{tX} \in G \, \forall \, t \in \mathbb{R} \}.$$

and the bracket [X, Y] of two elements of \mathfrak{g} is equal to the element XY - YX of $M_{n \times n}(\mathbb{C})$.

Using the fact that $\det(e^X) = e^{\operatorname{tr} X}$, we can see that the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, resp. $\mathfrak{sl}_n(\mathbb{R})$, of $SL_n(\mathbb{C})$, resp. $SL_n(\mathbb{R})$, is just the set of matrices in $M_{n \times n}(\mathbb{C})$, resp. $M_{n \times n}(\mathbb{R})$, that have trace equal to 0. Now suppose that $G = Sp_{2n}(F)$ with $F = \mathbb{R}$ or \mathbb{C} . Note that $e^{tX} \in G$ if and only if $Je^{tX^t}J^{-1} = e^{-tX}$. From

$$X = \frac{d}{dt}(e^{tX})|_{t=0} = \lim_{t \to 0} (e^{tX} - 1)/t.$$

we can see that $Je^{tX^t}J^{-1} = e^{-tX}$ for all $t \in \mathbb{R}$ implies $JX^tJ^{-1} = -X$. The converse is easy to see. Therefore the Lie algebra $\mathfrak{sp}_{2n}(F)$ of $Sp_{2n}(F)$ is given by

$$\mathfrak{sp}_{2n}(F) = \{ X \in M_{n \times n}(F) \mid JX^t J^{-1} = -X \} = \{ X \in M_{n \times n}(F) \mid JX^t + XJ = 0 \}.$$

The same type of approach can be used to find the Lie algebras of orthogonal and unitary matrix Lie groups.

If V is a finite-dimensional complex vector space, $\operatorname{End}_{\mathbb{C}}(V)$ is a Lie algebra relative to the bracket $[X, Y] = X \circ Y - Y \circ X$. This Lie algebra is denoted by $\mathfrak{gl}(V)$. A linear map $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism if

$$\phi([X,Y]) = \phi(X) \circ \phi(Y) - \phi(Y) \circ \phi(X), \qquad X, Y \in \mathfrak{g}.$$

A finite-dimensional representation of \mathfrak{g} is a Lie algebra homomorphism from \mathfrak{g} to $\mathfrak{gl}(V)$ for some finite-dimensional complex vector space V.

Proposition. Let G be a matrix Lie group, and let (π, V) be a continuous finite-dimensional representation of G. Then there is a unique representation $d\pi$ of the Lie algebra \mathfrak{g} of G (acting on the space V) such that

$$\pi(e^X) = e^{d\pi(X)}, \qquad X \in \mathfrak{g}.$$

Furthermore $d\pi(X) = \frac{d}{dt}\pi(e^{tX})|_{t=0}$, $X \in \mathfrak{g}$, and π is irreducible if and only if $d\pi$ is irreducible.

If the matrix Lie group G is simply connected, that is, the topological space G is simply connected, then any finite-dimensional representation of \mathfrak{g} lifts to a finite-dimensional representation of G, and the representations of G and \mathfrak{g} are related as in the above proposition.

5.6. Groups of t.d. type

A Hausdorff topological group is a t.d. group if G has a countable neighbourhood basis at the identity consisting of compact open subgroups, and G/K is a countable set for every open subgroup K of G. Some t.d. groups are matrix groups over p-adic fields.

Let p be a prime. Let $x \in \mathbb{Q}^{\times}$. Then there exist unique integers m, n and r such that m and n are nonzero and relatively prime, p does not divide m or n, and $x = p^r m/n$. Set $|x|_p = p^{-r}$. This defines a function on \mathbb{Q}^{\times} , which we extend to a function from \mathbb{Q} to the set of nonnegative real numbers by setting $|0|_p = 0$. The function $|\cdot|_p$ is called the *p*-adic absolute value on \mathbb{Q} . It is a valuation on \mathbb{Q} - that is, it has the properties

- (i) $|x|_p = 0$ if and only if x = 0
- (ii) $|xy|_p = |x|_p |y|_p$
- (iii) $|x+y|_p \le |x|_p + |y|_p$.

The usual absolute value on the real numbers is another example of a valuation on \mathbb{Q} . The *p*-adic abolute value satisfies the *ultrametric* inequality, that is, $|x + y|_p \leq \max\{|x|_p, |y|_p\}$. Note that the ultrametric inequality implies property (iii) above. A valuation that satisfies the ultrametric inequality is called a *nonarchimedean* valuation. Note that the set $\{ |x|_p \mid x \in \mathbb{Q}^{\times} \}$ is a discrete subgroup of \mathbb{R}^{\times} . Hence we say that $|\cdot|_p$ is a *discrete* valuation. The usual absolute value on \mathbb{Q} is an example of an *archimedean* valuation. Clearly it is not a discrete valuation. Two valuations on a field F are said to be *equivalent* is one is a positive power of the other.

Theorem. (Ostrowski) A nontrivial valuation on \mathbb{Q} is equivalent to the usual absolute value or to $|\cdot|_p$ for some prime p.

If F is a field and $|\cdot|$ is a valuation on F, the topology on F induced by $|\cdot|$ has as a basis the sets of the form $U(x, \epsilon) = \{y \in F \mid |x - y| < \epsilon\}$, as x varies over F, and ϵ varies over all positive real numbers. A field F' with valuation $|\cdot|'$ is a completion of the field F with valuation $|\cdot|$ if $F \subset F'$, |x|' = |x| for all $x \in F$, F' is complete with respect to $|\cdot|'$ (every Cauchy sequence with respect to $|\cdot|'$ has a limit in F') and F' is the closure of F with respect to $|\cdot|$. So F' is the smallest field containing F such that F' is complete with respect to $|\cdot|'$.

The real numbers is the completion of \mathbb{Q} with respect to the usual absolute value on \mathbb{Q} .

The *p*-adic numbers \mathbb{Q}_p is the completion of \mathbb{Q} with respect to $|\cdot|_p$. (We denote the extension of $|\cdot|_p$ to \mathbb{Q}_p by $|\cdot|_p$ also). The *p*-adic integers \mathbb{Z}_p is the set $\{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$. Note that \mathbb{Z}_p is a subring of \mathbb{Q}_p (this follows from the ultrametric inequality and the multiplicative property of $|\cdot|_p$), and \mathbb{Z}_p contains \mathbb{Z} . The set $p\mathbb{Z}_p$ (the ideal of \mathbb{Q}_p generated by the element p) is a maximal ideal of \mathbb{Z}_p and $\mathbb{Z}_p/p\mathbb{Z}_p$ is therefore a field.

Let $x \in \mathbb{Q}^{\times}$. Write $x = p^r m/n$ with $r \in \mathbb{Z}$ and m and n nonzero integers such that mand n are relatively prime and not divisible by p. Because m and n are relatively prime and not divisible by p, the equation $nX \equiv m \pmod{p}$ has a unique solution $a_r \in \{1, \ldots, p-1\}$. That is, there is a unique integer $a_r \in \{1, \ldots, p-1\}$ such that p divides $m - na_r$. Since $|n|_p = 1, p$ divides $m - na_r$ is equivalent to $|(m/n) - a_r|_p < 1$, and also to $|x - a_r p^r|_p <$ $|x|_p = p^{-r}$. Expressing $x - a_r p^r$ in the form $p^s m'/n'$ with s > r and m' and n' relatively prime integers, we repeat the above argument to produce an integer $a_s \in \{1, \ldots, p-1\}$ such that

$$|x - a_r p^r - a_s p^s|_p < |x - a_r p^r|_p = p^{-s}.$$

If s > r+1, set $a_{r+1} = a_{r+2} = \cdots = a_{s-1} = 0$, to get

$$|x - \sum_{n=r}^{s} a_n p^n|_p < p^{-s}$$

Continuing in this manner, we see that there exists a sequence $\{a_n \mid n \geq r\}$ such that $a_n \in \{0, 1, \ldots, p-1\}$ and, given any integer $M \geq r$,

$$|x - \sum_{n=r}^{M} a_n p^n|_p < p^{-M}$$

It follows that $\sum_{n=r}^{\infty} a_n p^n$ converges in the *p*-adic topology to the rational number *x*.

On the other hand, it is quite easy to show that if $a_n \in \{0, 1, ..., p-1\}$ and r is an integer, then $\sum_{n=r}^{\infty} a_n p^n$ converges to an element of \mathbb{Q}_p (though not necessarily to a rational number).

Lemma. A nonzero element x of \mathbb{Q}_p is uniquely of the form $\sum_{n=r}^{\infty} a_n p^n$, with $a_n \in \{0, 1, 2, \ldots, p-1\}$, for some integer r with $a_r \neq 0$. Furthermore, $|x|_p = p^{-r}$. (Hence $x \in \mathbb{Z}_p$ if and only if $r \geq 0$).

Lemma. $\mathbb{Z}_p/p\mathbb{Z}_p \simeq \mathbb{Z}/p\mathbb{Z}$.

Proof. Let $a \in \mathbb{Z}_p$. According to the above lemma, $a = \sum_{n=r}^{\infty} a_n p^n$ for some sequence $\{a_n \mid n \geq r\}$, where $|a|_p = p^{-r} \leq 1$ implies that $r \geq 0$. If r > 0, then $a \in p\mathbb{Z}_p$. For convenience, set $a_0 = 0$ when $|a|_p < 1$. If r = 0, then $a_0 \in \{1, \ldots, p-1\}$. Define a map from \mathbb{Z}_p to $\mathbb{Z}/p\mathbb{Z}$ by $a \mapsto a_0$. This is a surjective ring homomorphism whose kernel is equal to $p\mathbb{Z}_p$. qed

A local field F is a (nondiscrete) field F which is locally compact and complete with respect to a nontrivial valuation. The fields \mathbb{R} , \mathbb{C} , and \mathbb{Q}_p , p prime, are local fields. If $|\cdot|$ is a nontrivial nonarchimedean valuation on a field F, then $\{x \in F \mid |x| < 1\}$ is a maximal ideal in the ring $\{x \in F \mid |x| \le 1\}$, so the quotient is a field, called the *residue class field* of F. The following lemma can be used to check that \mathbb{Q}_p is a local field.

Lemma. Let $|\cdot|$ be a nonarchimedean valuation on a field F. Then F is locally compact with respect to $|\cdot|$ if and only if

- (1) F is complete (with respect to $|\cdot|$)
- (2) $|\cdot|$ is discrete
- (3) The residue class field of F is finite.

For every integer N, $p^N \mathbb{Z}_p$ is a compact open (and closed) subgroup of \mathbb{Q}_p . It is not hard to see that $\{p^N \mathbb{Z}_p \mid N \ge 0\}$ forms a countable neighbourhood basis at the identity element 0. From the above lemma, we have that $\mathbb{Q}_p^{\times} \simeq \langle p \rangle \times \mathbb{Z}_p^{\times}$. Hence $\mathbb{Q}_p/\mathbb{Z}_p$ is discrete. It can be shown that any open subgroup of \mathbb{Q}_p is of the form $p^N \mathbb{Z}_p$ for some integer N. Thus \mathbb{Q}_p/K is discrete for every open subgroup K of \mathbb{Q}_p . So the group \mathbb{Q}_p is a t.d. group. For more information on valuations, the *p*-adic numbers, and *p*-adic fields, see the beginning of [M] (course notes for Mat 1197).

As discussed in the section 5.3, because \mathbb{Q}_p is locally compact, the topological group $GL_n(\mathbb{Q}_p)$ is also locally compact. In fact $GL_n(\mathbb{Q}_p)$ is a t.d. group. If j is a positive integer, let K_j be the set of $g \in GL_n(\mathbb{Q}_p)$ such that every entry of g-1 belongs to $p^j\mathbb{Z}_p$. Then K_j is a compact open subgroup, and $\{K_j \mid j \geq 1\}$ forms a countable neighbourhood basis at the identity element 1.

Let K be an open subgroup of $GL_n(\mathbb{Q}_p)$. Then $K_j \subset K$ for some $j \geq 1$. Hence to prove that G/K is countable, it suffices to prove that G/K_j is countable for every j. For a discussion of the proof that G/K_j is countable, see [M].

Closed subgroups (and open subgroups) of t.d. groups are t.d. So any closed subgroup of $GL_n(\mathbb{Q}_p)$ is a t.d. group. These groups are often called *p*-adic groups. We remark that groups like $GL_n(\mathbb{Q}_p)$, $SL_n(\mathbb{Q}_p)$, $Sp_{2n}(\mathbb{Q}_p)$, etc., are the groups of \mathbb{Q}_p -rational points of reductive linear algebraic groups that are defined over \mathbb{Q}_p . Such groups have another topology, the Zariski topology (coming from the variety that is the algebraic group). The structure of these groups is often studied via algebraic geometry, in contrast with the structure of Lie groups, which is studied via differential geometry.

As with Lie groups, there is a notion of smoothness for representations. A (complex) representation (π, V) is *smooth* if for each $v \in G$, the subgroup $\{g \in G \mid \pi(g)v = v\}$ is an open subgroup of G. This definition is also valid if V is infinite-dimensional. This notion of smoothness is very different from that for Lie groups - in fact, connected Lie groups don't have any proper open subgroups. Because of the abundance of compact open subgroups in t.d. groups, and the fact that the general theory of representations of compact groups is well understood (see Chapter 6), properties of representations of t.d. groups are often studied via their restrictions to compact open subgroups.

Lemma. Suppose that (π, V) is a smooth finite-dimensional representation of a compact t.d. group G. Then there exists an open compact normal subgroup K of G and a representation ρ of the finite group G/K such that $\rho(gK)v = \pi(g)v$ for all $g \in G$ and $v \in K$.

Proof. By smoothness of π and finite-dimensionality of π , there exists an open compact subgroup K' of G such that $\pi(k')v = v$ for all $k' \in K'$ and $v \in V$. Choose a set $\{g_1, \ldots, g_r\}$ of coset representatives for G/K'. The subgroup $K := \bigcap_{j=1}^r k_j K' k_j^{-1}$ is an open compact normal subgroup of G and $\pi(k)v = v$ for all $k \in K$ and $v \in V$. It follows that there exists a representation (ρ, V) of the finite group G/K such that $\rho(gK)v = \pi(g)v, v \in V$, and $g \in G$. qed

We remark that the subgroup K' (hence the representation ρ) in the above lemma are not unique. Now suppose that (π, V) is a smooth (not necessarily finite-dimensional) representation of a (not necessarily compact) t.d. group G. Let K be a compact open subgroup of G. The restriction $\pi_K = r_G^K \pi$ of π to K is a (possibly infinite) direct sum of irreducible smooth representations of K. As we will see in Chapter 6, irreducible unitary representations of compact groups are finite-dimensional. Applying the above lemma, we can see that each of the irreducible representations of K which occurs in π_K is attached to a representation of some finite group. One difficulty in studying the representations of non-compact t.d. groups involves determining which compact open subgroups K and which irreducible constituents of π_K can be used to effectively study properties of π . For more information on representations of p-adic groups, see the notes [C] or the course notes for Mat 1197 ([M])

5.7. Haar measure on locally compact groups

If X is a topological space, a σ -ring in X is a nonempty family of subsets of X having the property that arbitrary unions of elements in the family belong to the family, and if A and B belong to the family, then so does $\{x \in A \mid x \notin B\}$. If X is a locally compact topological space, the *Borel ring* in X is the smallest σ -ring in X that contains the open sets. The elements of the Borel ring are called *Borel sets*. A function $f: X \to \mathbb{R}$ is *(Borel) measurable* if for every t > 0, the set $\{x \in X \mid |f(x)| < t\}$ is a Borel set.

Let G be a locally compact topological group. A left Haar measure on G is a nonzero regular measure μ_{ℓ} on the Borel σ -ring in G that is left G-invariant: $\mu_{\ell}(gS) = \mu_{\ell}(S)$ for measurable set S and $g \in G$. Regularity means that

 $\mu_{\ell}(S) = \inf\{ \mu_{\ell}(U) \mid U \supset S, U \text{ open} \} \text{ and } \mu_{\ell}(S) = \sup\{ \mu_{\ell}(C) \mid C \subset S, C \text{ compact} \}.$

Such a measure has the properties that any compact set has finite measure and any nonempty open set has positive measure. Left invariance of μ_{ℓ} amounts to the property

$$\int_G f(g_0 g) d\mu_\ell(g) = \int_G f(g) d\mu_\ell(g), \qquad \forall \ g_0 \in G,$$

for any Haar integrable function f on G.

Theorem. ([Halmos], [HR], [L]) If G is a locally compact group, there is a left Haar measure on G, and it is unique up to positive real multiples.

There is also a right Haar measure μ_r , unique up to positive constant multiples, on G. Right and left Haar measures do not usually coincide.

Exercise. Let

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R}^{\times}, y \in \mathbb{R} \right\}.$$

Show that $|x|^{-2}dx dy$ is a left Haar measure on G and $|x|^{-1}dx dy$ is a right Haar measure on G.

The (locally compact topological) group G is called *unimodular* if each left Haar measure is also a right Haar measure. Clearly, G is unimodular if G is abelian. Conjugation by a fixed $g_0 \in G$ is a homemomorphism of G onto itself, so the measure $S \mapsto \mu_{\ell}(g_0 S g_0^{-1}) =$ $\mu_{\ell}(S g_0^{-1})$ (S measurable) is also a left Haar measure. By uniqueness of left Haar measure, there exists a constant $\delta(g_0) > 0$

$$\int_G f(g_0 g g_0^{-1}) d\mu_\ell(g) = \delta(g_0) \int_G f(g) d\mu_\ell(g), \qquad f \text{ integrable}$$

A quasicharacter of G is a continuous homomorphism from G to \mathbb{C}^{\times} .

Proposition.

- (1) The function $\delta: G \to \mathbb{R}^{\times}_+$ is a quasicharacter
- (2) $\delta(g)d\mu_{\ell}(g)$ is a right Haar measure.

Proof. The fact that conjugation is an action of G on itself implies that $\delta: G \to \mathbb{R}_+^{\times}$ is a homomorphism. The proof of continuity is omitted. Note that

$$\delta(g_0) \int_G f(g) \, d\mu_\ell(g) = \int_G f(g_0 \cdot g_0^{-1} g g_0) \, d\mu_\ell(g) = \int_G f(g g_0) \, d\mu_\ell(g).$$

Replacing f by $f\delta$ and dividing both sides by $\delta(g_0)$, we obtain

$$\int_G f(g)\delta(g) \, d\mu_\ell(g) = \int_G f(gg_0)\delta(g) \, d\mu_\ell(g).$$

This shows that $\delta(g)d\mu_{\ell}(g)$ is right invariant. qed

In view of the above, we may write $d\mu_r(g) = \delta(g)d\mu_\ell(g)$. The function δ is called the *modular quasicharacter* of G. Clearly G is unimodular if and only if the modular quasicharacter is trivial. If G is unimodular, we simply refer to Haar measure on G.

Exercises:

- (1) Let dX denote Lebesgue measure on $M_{n \times n}(\mathbb{R})$. This is a Haar measure on $M_{n \times n}(\mathbb{R})$. Show that $|det(g)|^{-n}dg$ is both a left and a right Haar measure on $GL_n(\mathbb{R})$. Hence $GL_n(\mathbb{R})$ is unimodular.
- (2) Let n_1 and n_2 be positive integers such that $n_1 + n_2 = n$. $P = P_{(n_1,n_2)}$ be the standard parabolic subgroup of $GL_n(\mathbb{R})$ corresponding to the partition (n_1, n_2) (see Chapter 4 for the definition of standard parabolic subgroup of a general linear group). Let $g = \begin{pmatrix} g_1 & X \\ 0 & g_2 \end{pmatrix} \in P$, with $g_j \in GL_{n_j}(\mathbb{R})$ and $X \in M_{n_1 \times n_2}(\mathbb{R})$. Let dg_j be Haar measure on $GL_{n_j}(\mathbb{R})$, and let dX be Haar measure on $M_{n_1 \times n_2}(\mathbb{R})$. Show that $d_{\ell}g = |\det g_1|^{-n_2}dg_1 dg_2 dX$ and $d_rg = |\det g_2|^{-n_1}dg_1 dg_2 dX$ are left and right Haar measures on P (respectively). Hence the modular quasicharacter of P is equal to $\delta(g) = |\det g_1|^{n_1} |\det g_2|^{-n_2}$.
- (3) Show that the homeomorphism $g \mapsto g^{-1}$ turns μ_{ℓ} into a right Haar measure. Conclude that if G is unimodular, then $\int_G f(g) d\mu_{\ell}(g) = \int_G f(g^{-1}) d\mu_{\ell}(g)$ for all measurable functions f.

Proposition. If G is compact, then G is unimodular and $\mu_{\ell}(G) < \infty$.

Proof. Since δ is a continuous homomorphism and G is compact, $\delta(G)$ is a compact subgroup of \mathbb{R}_+^{\times} . But {1} is the only compact subgroup of \mathbb{R}_+^{\times} . Haar measure on any locally compact group has the property that any compact subset has finite measure. Hence $\mu_{\ell}(G) < \infty$ whenever G is compact. qed If G is compact, normalized Haar measure on G is the unique Haar measure μ on G such that $\mu(G) = 1$. When working with compact groups, we will always work relative to normalized Haar measure and we will write write $\int_G f(g) dg$ for $\int_G f(g) d\mu(g)$.

5.8. Discrete series representations

Let G be a locally compact unimodular topological group. A unitary representation π of G on a Hilbert space V (with inner product $\langle \cdot, \cdot \rangle$) is *continuous* if for every $v, w \in V$, the function $g \mapsto \langle \pi(g)v, w \rangle$ is a continuous function on G. That is, matrix coefficients of π are continuous functions on G. Note that such a representation may be infinite-dimensional. (In particular, if G is a noncompact semisimple Lie group, then all nontrivial irreducible continuous unitary representations of G are infinite-dimensional.)

Suppose that (π, V) is an irreducible continuous unitary representation of G. Let Z be the centre of G. A generalization of Schur's Lemma to this setting shows that if $z \in Z$, then there exists $\omega(z) \in \mathbb{C}^{\times}$ such that $\pi(z) = \omega(z)I$. Because π is a continuous unitary representation, the function $z \mapsto \omega(z)$ is a continuous linear character of the group Z. In particular, $|\omega(z)| = 1$ for all $z \in Z$. The representation π is said to be *square-integrable* mod Z, or to be a *discrete series* representation, if there exist nonzero vectors v and $w \in V$ such that

$$\int_{G/Z} |\langle v, \pi(g)w\rangle|^2 \, dg^{\times} < \infty,$$

where dg^{\times} is Haar measure on the locally compact group G/Z. Thus π is a discrete series representation if some nonzero matrix coefficient of π is square-integrable modulo Z.

Fix an ω as above. Let $C_c(G, \omega)$ be the space of continuous functions from $f \ G$ to \mathbb{C} that satisfy $f(zg) = \omega(z)f(g)$ for all $g \in G$ and $z \in Z$, and are compactly supported modulo Z (there exists a compact subset C_f of G such that the support of f lies inside the set $C_f Z$). Define an inner product on $C_c(G, \omega)$ by $(f_1, f_2) = \int_{G/Z} f_1(g)\overline{f_2(g)} dg^{\times}$. Let $L^2(G, \omega)$ be the completion of $C_c(G, \omega)$ relative to the norm $||f|| = (f, f)^{1/2}, f \in C_c(G, \omega)$. The group G acts by right translation on $L^2(G, \omega)$, and this defines a continuous unitary representation of G on the Hilbert space $L^2(G, \omega)$.

Theorem. (Schur orthogonality relations). Let (π, V) and (π', V') be irreducible continuous unitary representations of G such that $\omega = \omega'$.

- (1) The following are equivalent:
 - (i) π is square-integrable mod Z.
 - (ii) $\int_{G/Z} |\langle v, \pi(g)w \rangle|^2 dg^{\times} < \infty$ for all $v, w \in V$.
 - (iii) π is equivalent to a subrepresentation of the right regular representation of G on $L^2(G, \omega)$.

(2) If the conditions of (1) hold, then there exists a number $d(\pi) > 0$, called the formal degree of π (depending only on the normalization of Haar measure on G/Z), such that

$$\int_{G/Z} \overline{\langle v_1, \pi(g)w_1 \rangle} \langle v_2, \pi(g)w_2 \rangle \, dg^{\times} = d(\pi)^{-1} \overline{\langle v_1, v_2 \rangle} \langle w_1, w_2 \rangle, \qquad \forall \ v_1, v_2, w_1, w_2 \in V.$$

(3) If π is not equivalent to π' , then

$$\int_{G/Z} \overline{\langle v, \pi(g)w \rangle} \langle v', \pi'(g)w' \rangle \, dg^{\times} = 0 \qquad \forall \ v, w \in V, \ v', w' \in V'$$

5.9. Parabolic subgroups and representations of reductive groups

The description of the parabolic subgroups of general linear groups and special linear groups over finite fields given in Chapter 4 is valid for general linear and special linear groups over any field F - simply replace the matrix entries in the finite field by matrix entries in the field F. General linear and special linear groups are examples of reductive groups. We do not give the definition of parabolic subgroup for arbitrary reductive groups.

Suppose that G is the F-rational points of a connected reductive linear algebraic group, where $F = \mathbb{R}$, $F = \mathbb{C}$, F is a p-adic field (for example, $F = \mathbb{Q}_p$), or F is a finite field.

The "Philosophy of Cusp Forms" says that the collection of representations of a reductive group G should be partitioned into disjoint subsets in such a way that each subset is attached to an associativity class of parabolic subgroups of G. Two parabolic subgroups $P = M \ltimes N$ and $P' = M' \ltimes N'$ of G are associate if and only if the Levi factors Mand M' are conjugate in G. The representations attached to the group G itself are called *cuspidal* representations, and their matrix coefficients are called *cusp forms*. If P is a proper parabolic subgroup of G, the representations attached to P are associated to (Weyl group orbits of) cuspidal representations of a Levi factor M of P. (Note that M is itself a reductive group). Furthermore, the representations of G associated to a given cuspidal representation σ of M occur as subquotients of the induced representation $\operatorname{Ind}_P^G(\sigma \otimes \delta_P^{1/2})$, where δ_P is the modular quasicharacter of P (see § 5.7) and σ is extended to a representation of $P = M \ltimes N$ by letting it be trivial on N.

The problem of understanding the representations of the group G can be approached via the Philosophy of Cusp Forms, and is therefore divided into two parts. The first part is to determine the cuspidal representations of the Levi subgroups M of G, and the second part is to analyze representations parabolically induced from such cuspidal representations.

In certain contexts, a cuspidal representation is simply a discrete series representation (see § 5.8 for the definition of discrete series representation). If G is a connected reductive Lie group (for example $G = SL_n(\mathbb{R})$ or $G = Sp_{2n}(\mathbb{R})$, then there are two cases to consider. An element of a matrix Lie group is *semisimple* if it is semsimple as a matrix, that is, it can be diagonalized over the field of complex numbers. A *Cartan subgroup* of G is a closed subgroup that is a maximal abelian subgroup consisting of semisimple elements. In the first case, G contains no Cartan subgroups that are compact modulo the centre of G (for example, this is the case if G is semisimple and $F = \mathbb{C}$, of if $SL_n(\mathbb{R})$ and $n \geq 3$), and hence G has no discrete series representations. In the second case, up to conjugacy G contains one Cartan subgroup T that is compact modulo the centre of G, and the discrete series of G are parametrized in a natural way by the so-called *regular* characters of T.

An irreducible unitary representation of G is *tempered* if it occurs in the decomposition of the regular representation of G on the Hilbert space $L^2(G)$ of square-integrable functions on G. If π is a tempered representation of G, then there exists a parabolic subgroup $P = M \ltimes N$ and a discrete series representation of M such that π occurs as a constituent of the induced representation $\operatorname{Ind}_P^G(\sigma \otimes \delta_P^{1/2})$.

If G is a reductive p-adic group (that is, F is a p-adic field), a continuous complexvalued function f on G is a supercusp form if the support of f is compact modulo the centre of G and $\int_N f(gn) dn = 0$ for all $g \in G$ and all unipotent radicals N of proper parabolic subgroups of G. An irreducible smooth representation (where a smooth representation is as defined in §5.6) of G is supercuspidal if the matrix coefficients of the representation are supercusp forms. Given an irreducible smooth representation π of G, there exists a parabolic subgroup $P = M \ltimes N$ of G and a supercuspidal representation σ of M such that π is a subquotient of $\operatorname{Ind}_P^G(\sigma \otimes \delta_P^{1/2})$. Hence in this context, it is suitable to interpret "cuspidal representation" as supercuspidal representation. (Recall that a similar result was described in Chapter 4 in the case that F is a finite field).

If π is a supercuspidal representation of G, then there exists a quasicharacter ω of the centre Z of G such that $\pi(z) = \omega(z)I$, $z \in Z$. It is easy to see that if ω is unitary (that is, $|\omega(z)| = 1$ for all $z \in Z$), then π is a discrete series representation. A reductive *p*-adic group has many supercuspidal representations and hence many discrete series representations. However, there exist discrete series representations that are not supercuspidal.