CHAPTER 4

Representations of finite groups of Lie type

Let $\mathbb{F}_q$ be a finite field of order q and characteristic p. Let G be a finite group of Lie type, that is, G is the $\mathbb{F}_q$-rational points of a connected reductive group $\mathbb{G}$ defined over $\mathbb{F}_q$. For example, if n is a positive integer $GL_n(\mathbb{F}_q)$ and $SL_n(\mathbb{F}_q)$ are finite groups of Lie type. Let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, where I_n is the $n \times n$ identity matrix. Let

$$Sp_{2n}(\mathbb{F}_q) = \{\, g \in GL_{2n}(\mathbb{F}_q) \mid {}^t gJg = J \,\}.$$

Then $Sp_{2n}(\mathbb{F}_q)$ is a symplectic group of rank n and is a finite group of Lie type.

For $G = GL_n(\mathbb{F}_q)$ or $SL_n(\mathbb{F}_q)$ (and some other examples), the *standard Borel subgroup* B of G is the subgroup of G consisting of the upper triangular elements in G. A *standard parabolic subgroup* of G is a subgroup of G which contains the standard Borel subgroup B. If P is a standard parabolic subgroup of $GL_n(\mathbb{F}_q)$, then there exists a partition $(n_1, \ldots, n_r)$ of n (a set of positive integers n_j such that $n_1 + \cdots + n_r = n$) such that $P = P_{(n_1,\ldots,n_r)} = M \ltimes N$, where $M \simeq GL_{n_1}(\mathbb{F}_q) \times \cdots \times GL_{n_r}(\mathbb{F}_q)$ has the form

$$M = \left\{ \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_r \end{pmatrix} \;\middle|\; A_j \in GL_{n_j}(\mathbb{F}_q),\ 1 \le j \le r \right\}.$$

and

$$N = \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{n_r} \end{pmatrix} \right\},$$

where $*$ denotes arbitary entries in $\mathbb{F}_q$. The subgroup M is called a (standard) Levi subgroup of P, and N is called the unipotent radical of P. Note that the partition $(1, 1, \ldots, 1)$ corresponds to B and (n) corresponds to G. A standard parabolic subgroup of $SL_n(\mathbb{F}_q)$ is equal to $P_{(n_1,\ldots,n_r)} \cap SL_n(\mathbb{F}_q)$ for some partition $(n_1, \ldots, n_r)$ of n. A *parabolic subgroup* of G is a subgroup which is conjugate to a standard parabolic subgroup. By replacing $\mathbb{F}_q$ by a field F in the above definitions, we can define parabolic subgroups of $GL_n(F)$. For more information on parabolic subgroups of general linear groups see the book of Alperin and Bell.

For convenience, assume that $G = GL_n(\mathbb{F}_q)$. Let $P = P_{(n_1,\ldots,n_r)}$ be a proper standard parabolic subgroup of G. Let π_j be an irreducible representation of $GL_{n_j}(\mathbb{F}_q)$, $1 \le j \le r$.

Then $\pi := \pi_1 \otimes \cdots \otimes \pi_r$ is an irreducible representation of M. Extend π to a representation of P by letting elements of N act via the identity (on the space of π). Let $i_P^G \pi = \mathrm{Ind}_P^G \pi$ be the representation of G induced from π. This process of going from a representation of a Levi subgroup of a proper parabolic subgroup to a representation of G is known as parabolic induction (or Harish-Chandra induction). It is possible to show that if P' is another parabolic subgroup of G having M as a Levi subgroup, and unipotent radical N', then $i_P^G \pi \simeq i_{P'}^G \pi$. For this reason, the notation $i_M^G \pi$ is sometimes used in place of $i_P^G \pi$.

Now we can define a standard parabolic subgroup P_M of M to be a group of the form $P_M = P_1 \times \cdots \times P_r$ where P_j is a standard parabolic subgroup of $GL_{n_j}(\mathbb{F}_q)$. Write $P_j = M_j \ltimes N_j$, where M_j is the standard Levi subgroup of P_j and N_j is the unipotent radical of P_j. Let $N_M = N_1 \times \cdots N_r$. Then $P_M N$ is a standard parabolic subgroup of $GL_{n_j}(\mathbb{F}_q)$, with standard Levi factor $M_1 \times \cdots \times M_r$ and unipotent radical $N_M N$. Let $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r$ where σ_j is an irreducible representation of M_j. Then we can check that $i_{P_M N}^G \sigma \simeq i_P^G(i_{P_M}^M \sigma)$, where on the left side, σ is extended to $P_M N$ by letting elements of $N_M N$ act trivially, and on the right side, σ is extended to P_M by letting N_M act trivially. This property is called transitivity of parabolic induction.

Let N be the unipotent radical of a proper parabolic subgroup of G and let π be a representation of G. Then the restriction $r_G^N \pi = \pi_N$ of π to N is a direct sum of irreducible representations of N. Via Frobenius reciprocity type arguments, we can prove that if an irreducible representation π of G contains the trivial representation of the unipotent radical of proper parabolic subgroup of G, then π occurs as a subrepresentation of a parabolically induced representation of G. An irreducible representation π of G is *cuspidal* if $r_G^N \pi$ does not contain the trivial representation of N for all choices of unipotent radicals of proper parabolic subgroups of G. If π is cuspidal, then by Frobenius reciprocity, $\mathrm{Hom}_G(\pi, i_P^G \pi') = 0$ if P is a proper parabolic subgroup of G and π' is an irreducible representation of a Levi factor of P. The following theorem is valid for irreducible representations of finite groups of Lie type, not just for general linear groups.

Theorem. *(Proposition 9.13 of Carter's book) Let π be an irreducible representation of G. Then one of the following holds:*

(1) π is cuspidal

(2) There exists a proper parabolic subgroup $P = M \ltimes N$ of G and a cuspidal representation π' of M such that $\mathrm{Hom}_G(\pi, i_P^G \pi') \neq 0$.

As a consequence of the above theorem, one approach to finding the irreducible representations of G involves two steps:

Step 1 : Find all cuspidal representations of those groups occurring as Levi factors of parabolic subgroups of G (including G itself).

Step 2 : Find all of the irreducible constituents of the representations $i_P^G \pi'$ where P is a proper parabolic subgroup of G and π' is a cuspidal representation of a Levi factor of P.

In $SL_2(\mathbb{F}_q)$, there is one conjugacy class of proper parabolic subgroups, namely the conjugacy class of the standard Borel subgroup. And the subgroup $A \simeq \mathbb{F}_q^\times$ of B is a standard Levi factor. Every irreducible representation (one-dimensional representation) of A is cuspidal, since A has no proper parabolic subgroups. The above theorem tells us that the non-cuspidal representations of $GL_2(\mathbb{F}_q)$ are exactly the irreducible subrepresentations of the representations $i_B^G\tau$ as τ ranges over the set of one-dimensional representations of A. As we saw in Chapter 3, letting ε be a fixed nonsquare in $\mathbb{F}_q^\times$, the cuspidal representations of $SL_n(\mathbb{F}_q)$ are associated to the nontrivial characters of the group

$$T = \left\{ \begin{pmatrix} a & b\varepsilon \\ b & a \end{pmatrix} \mid a,\, b \in \mathbb{F}_q,\ a^2 - b^2\varepsilon = 1 \right\}.$$

In the case of $SL_2(\mathbb{F}_q)$ the groups A and T are representatives for the conjugacy classes of maximal tori in $SL_2(\mathbb{F}_q)$.

In 1976, for G a finite group of Lie type, Deligne and Lusztig published a paper showing that it is possible to associate a virtual character of G to each character of a maximal torus (see references).

Next we describe the maximal tori in $GL_n(\mathbb{F}_q)$ and $SL_n(\mathbb{F}_q)$. Let k be a positive integer. The $\mathbb{F}_{q^k}$ is a vector space over $\mathbb{F}_q$ of dimension k. Choose a basis $\beta = \{\, x_1, \ldots, x_k \,\}$ of $\mathbb{F}_{q^k}$ over $\mathbb{F}_q$. Given $y \in \mathbb{F}_{q^k}$, let $g_y \in GL_k(\mathbb{F}_q)$ be the matrix relative to β of the linear transformation on $\mathbb{F}_{q^k}$ given by left multiplication by y. Then $\{\, g_y \mid y \in \mathbb{F}_{q^k}^\times \,\}$ is a subgroup of $GL_k(\mathbb{F}_q)$ which is isomorphic, via $y \mapsto g_y$, to $\mathbb{F}_{q^k}^\times$. This subgroup depends on the choice of basis β. Any two such subgroups of $GL_k(\mathbb{F}_q)$ are conjugate, via the relevant change of basis matrix.

Suppose that $n_1, \ldots, n_r$ are integers such that $n_1 \geq n_2 \geq \cdots \geq n_r > 0$ and $n_1 + \cdots + n_r = n$. Fix a basis of $\mathbb{F}_{q^{n_j}}$ over $\mathbb{F}_q$, $1 \leq j \leq r$. Then the group $\mathbb{F}_{q^{n_1}}^\times \times \cdots \times \mathbb{F}_{q^{n_r}}^\times$, is isomorphic to a subgroup $T_{(n_1,\ldots,n_r)}$ of $GL_{n_1}(\mathbb{F}_q) \times \cdots \times GL_{n_r}(\mathbb{F}_q) \subset GL_n(\mathbb{F}_q)$. A *maximal torus* T in $GL_n(\mathbb{F}_q)$ is a subgroup which is conjugate to $T_{(n_1,\ldots,n_r)}$ for some $(n_1, \ldots, n_r)$ as above. A maximal torus in $SL_n(\mathbb{F}_q)$ is conjugate to some $T_{(n_1,\ldots,n_r)} \cap SL_n(\mathbb{F}_q)$. For $SL_2(\mathbb{F}_q)$ there are 2 conjugacy classes of maximal tori, represented by $T_{(1,1)} \cap SL_2(\mathbb{F}_q) = A$, and $T_{(2)} \cap SL_2(\mathbb{F}_q)$, the group T mentioned above.

Let T be a maximal torus in G and let θ be a character of T. Let $R_{T,\theta}$ be the virtual character of G associated to the pair (T, θ) by Deligne and Lusztig. The value of $R_{T,\theta}$ on an element g of G is given by an alternating sum of the trace of the action of g on certain ℓ-adic cohomology groups. For more information on this, see the paper of Deligne and Lusztig, or the book of Carter.

A matrix $u \in GL_n(\mathbb{F}_q)$ is *unipotent* if $u - 1$ is nilpotent. A matrix $\gamma \in GL_n(\mathbb{F}_q)$ is *semisimple* if γ is diagonalizable over some finite extension of $\mathbb{F}_q$. Let $g \in GL_n(\mathbb{F}_q)$. The characteristic polynomial of g, being a polynomial of degree n, splits over a finite extension $\mathbb{F}_{q^k}$ of $\mathbb{F}_q$, for some $k \leq n$. Results from linear algebra tell us that there exist matrices γ and $u \in GL_n(\mathbb{F}_{q^k})$ such that γ is diagonalizable and u is unipotent, with $\gamma u = u\gamma = g$. It can be shown that γ, $u \in GL_n(\mathbb{F}_q)$. Hence any element in $GL_n(\mathbb{F}_q)$ can be expressed in a unique way in the form $g = \gamma u$, with $\gamma \in GL_n(\mathbb{F}_q)$ semisimple, $u \in GL_n(\mathbb{F}_q)$ unipotent, and $\gamma u = u\gamma$. This is called the (multiplicative) *Jordan decomposition* of g. If G is a finite group of Lie type, then G is a subgroup of $GL_n(\mathbb{F}_q)$ for some n, and if $g = \gamma u$ is the Jordan decomposition of g in $GL_n(\mathbb{F}_q)$, the elements γ and u lie in G. So there is a Jordan decomposition for elements of a finite group of Lie type.

Attached to any a maximal torus T in G, there exists a particular class function Q_T^G on G, called the *Green function* corresponding to T. The Green function Q_T^G is supported on the unipotent set. If $G = GL_n(\mathbb{F}_q)$, the values of the Green functions are known. In fact, if u is unipotent the value $Q_T^G(u)$ is obtained as the value of a certain polynomial (depending on T and $\mathrm{cl}_G(u)$) in the variable q.

Theorem (Character formula for $R_{T,\theta}$). *Let T be a maximal torus in $G = GL_n(\mathbb{F}_q)$ or $SL_n(\mathbb{F}_q)$ and let θ be a character of T. Let $g \in GL_n(\mathbb{F}_q)$ have Jordan decomposition $g = \gamma u$, with semisimple part γ and unipotent part u. Let H be the centralizer of γ in G. Then*

$$R_{T,\theta}(g) = |H|^{-1} \sum_{x \in G,\, x^{-1}\gamma x \in T} \theta(x^{-1}\gamma x) Q^H_{xTx^{-1}}(u).$$

Exercises: Let $\gamma \in GL_n(\mathbb{F}_q)$ be semisimple.

(1) Prove that some conjugate of γ lies in $T_{(n_1,\ldots,n_r)}$ for some (not necessarily unique) $(n_1, \ldots, n_r)$.

(2) Prove that the centralizer H of γ in G is isomorphic to a direct product of the form $GL_{r_1}(\mathbb{F}_{q^{s_1}}) \times \cdots \times GL_{r_t}(\mathbb{F}_{q^{s_t}})$, where $r_1 s_1 + \cdots + r_t s_t = n$.

The character formula for $R_{T,\theta}$ resembles the Frobenius character formula in certain ways. If the conjugacy class of the semisimple part γ of g does not intersect T, then $R_{T,\theta}(g) = 0$. And when the conjugacy class of γ does intersect T, the expression for $R_{T,\theta}(g)$ involves values of θ on certain conjugates of γ in T. In the special case where $T = T_{(1,\ldots,1)}$ or $T_{(1,\ldots,1)} \cap SL_n(\mathbb{F}_q)$, then it can be shown that $R_{T,\theta} = \pm\chi_{i_B^G\theta}$.

If T and T' are maximal tori in G, let

$$N(T,T') = \{\, g \in G \mid gTg^{-1} = T' \,\}$$
$$W(T,T') = \{\, Tg \mid g \in N(T,T') \,\}$$

Theorem (Orthogonality relations for $R_{T,\theta}$'s). *Let θ and θ' be characters of T and T', respectively. Then*

$$\langle R_{T,\theta}, R_{T',\theta} \rangle = \{\, \omega \in W(T,T') \mid {}^{\omega}\theta' = \theta \,\},$$

where ${}^{\omega}\theta'(\gamma) := \theta'(g\gamma g^{-1})$, *for* $\gamma \in T$ *and* $\omega = Tg$.

Corollary. *If* T *and* T' *are not* G*-conjugate, then* $\langle R_{T,\theta}, R_{T',\theta'}\rangle = 0$.

Note that this corollary does not tell us that the set of irreducible characters appearing in $R_{T,\theta}$ is disjoint from those appearing in $R_{T',\theta'}$ when T and T' are not conjugate. Consider the example of $G = SL_2(\mathbb{F}_q)$. Let τ be a character of the maximal torus A in $G = SL_2(\mathbb{F}_q)$. According to comments above, $R_{A,\tau} = i_B^G\tau$. Let T be as in Chapter 3. Then it is possible to show that if θ is a character of T and φ_θ is as in Chapter 3, then $R_{T,\theta} = \pm\varphi_\theta$. Now suppose that $\tau = \tau_0$ is the trivial character of A and $\theta = \theta_0$ is the trivial character of T. Let χ_q be the character of the irreducible representation of $SL_2(\mathbb{F}_q)$ of degree q. Let χ_0 be the character of the trivial representation. As we saw in Chapter 3, $\chi_{i_B^G\tau_0} = R_{A,\tau_0} = \chi_0 + \chi_q$ and $\varphi_\theta = -\chi_0 + \chi_q$.

Let $N_G(T) = \{\, g \in G \mid gTg^{-1} = T \,\}$ be the normalizer of T in G. Then $W(T) := N_G(T)/T$ is a finite group. A character θ of T is said to be in *general position* if $\theta^w \neq \theta$ for all nontrivial elements of $W(T)$.

Theorem (corollary of above theorem). *If* θ *is in general position, then* $\pm R_{T,\theta}$ *is an irreducible character of* G.

Theorem. *If* π *is an irreducible representation of* G*, then there exists a maximal torus* T *of* G *and a character* θ *of* T *such that* $\langle \chi_\pi, R_{T,\theta}\rangle \neq 0$.

Suppose that $(n_1, \dots, n_r) \neq (n)$. Then $T_{(n_1,\dots,n_r)}$ is a subgroup of the Levi factor $M = GL_{n_1}(\mathbb{F}_q) \times \cdots \times GL_{n_r}(\mathbb{F}_q)$ of a proper parabolic subgroup P of G. Any class function f on M can be extended to a class function on $P = M \ltimes N$ by setting $f(mu) = f(m)$, $m \in M$, $u \in N$. Hence we can view i_P^G as a map from class functions on M to class functions on G. Deligne and Lusztig proved that if $T = T_{(n_1,\dots,n_r)}$ and θ is a character of T, then $R_{T,\theta} = i_P^G(R^M_{T,\theta})$, where $R^M_{T,\theta}$ is the virtual character of M corresponding to the pair (T, θ). (Note that T is a maximal torus in M). It follows that $R_{T,\theta}$ is in the span of the irreducible characters of G which occur as constituents of representations $i_P^G\sigma$, for various representations σ of M. Thus $\langle \chi_\pi, R_{T,\theta}\rangle = 0$ for all irreducible cuspidal representations π.

Proposition. *Suppose that* π *is an (irreducible) cuspidal representation of* $GL_n(\mathbb{F}_q)$, $n \geq 2$*. Then*

(1) $\langle \chi_\pi, R_{T_{(n)},\theta}\rangle \neq 0$ *for some character* θ *of* $T_{(n)}$.

(2) $\chi_\pi = \pm R_{T_{(n)},\theta}$ *for some character* θ *of* $T_{(n)}$ *that is in general position.*

(3) If T *is a maximal torus which is not conjugate to* $T_{(n)}$ *and* θ *is a character of* T*, then* $\langle \chi_\pi, R_{T,\theta}\rangle = 0$.

Parts (1) and (3) have analogues for cuspidal representations for other finite groups of Lie type. In those cases, the maximal torus $T_{(n)}$ must be replaced by a set of representatives for the conjugacy classes of maximal tori in G that do no intersect the Levi factor of any proper parabolic subgroup of G. The analogue of Part (2) does not hold for all cuspidal representations of other finite groups of Lie type. The two irreducible characters of of $SL_2(\mathbb{F}_q)$ of degree $(q-1)/2$ which were produced in Chapter 3 correspond to cuspidal representations whose characters are not of the form $\pm R_{T,\theta}$ (for any choice of T and θ). All of the irreducible characters of $SL_2(\mathbb{F}_q)$ of degree $q-1$ correspond to cuspidal representations whose characters equal $\pm R_{T,\theta}$ (with $\theta^2 \neq 1$, and T as in Chapter 3).

Hecke algebras are often used in the study of the representations of finite groups of Lie type. Suppose that P and P' are parabolic subgroups of G and (π, V) and (π', V') are irreducible representations of the Levi factors M and M' of standard parabolic subgroups P and P', respectively. Then, according to results from Chapter 2, the space $\mathrm{Hom}_G(i_P^G\pi, i_{P'}^G\pi')$ is isomorphic to the space of functions $\varphi : G \longrightarrow \mathrm{End}_{\mathbb{C}}(V, V')$ such that $\varphi(xgx') = \pi(x) \circ \varphi(g) \circ \pi'(x')$ for all $g \in G$, $x \in P$, and $x' \in P'$. In view of results discussed above, the case where π and π' are cuspidal is of particular interest. The following theorem is proved by studying properties of the above isomorphism - in particular, by analyzing the properties of the functions supported on each double coset PgP' and satisfying the above conditions.

Theorem. *Let π and π' be cuspidal representations of Levi factors M and M' of standard parabolic subgroups P and P' of $GL_n(\mathbb{F}_q)$, respectively. Then*

(1) If M and M' are not conjugate, then $\mathrm{Hom}_G(i_P^G\pi, i_{P'}^G\pi') = 0$.

(2) If M and M' are conjugate, then either $\mathrm{Hom}_G(i_P^G\pi, i_{P'}^G\pi') = 0$ or $i_P^G\pi \simeq i_{P'}^G\pi'$.

Suppose that $P' = P$ and $\pi' = \pi$. Then $\mathcal{H}(G, \pi)$ is a Hecke algebra which is isomorphic (as an algebra) to $\mathrm{Hom}_G(i_P^G\pi, i_P^G\pi)$. It is possible to prove that decomposition of representations of the form $i_P^G\pi$ into direct sums of irreducible representations can be reduced to cases where $P = P_{(m,\dots,m)}$ and $m \neq n$ is a divisor of n (with m occurring n/m times). In those cases, $\pi = \sigma \otimes \cdots \otimes \sigma$ for some cuspidal representation σ of $GL_m(\mathbb{F}_q)$ (where σ occurs n/m times in the tensor product). Let $G' = GL_{n/m}(\mathbb{F}_{q^m})$. Let B' be the standard Borel subgroup of G'. Then, letting π' be the trivial representation of G', we know that the Hecke algebra $\mathcal{H}(G', \pi')$ is isomorphic to $\mathrm{Hom}_{G'}(i_{B'}^{G'}\pi', i_{B'}^{G'}\pi')$.

Theorem. *(Notation as above). The algebra $\mathcal{H}(G, \pi)$ is isomorphic (in a canonical way) to the Hecke algebra $\mathcal{H}(G', \pi')$.*

Corollary. *There is a canonical bijection $\tau \leftrightarrow \tau'$ between the set of irreducible constituents τ of $i_P^G\pi$ and the set of irreducible constituents of τ' of $i_{B'}^{G'}\pi'$. The bijection satisifes:*

(1) The multiplicity of π in $r_G^P\tau$ equals the multiplicity of the trivial representation π' of B' in $r_{G'}^{B'}\tau$.

(2) The degree of τ *divided by the degree of* τ' *equals the degree of* π *times* $|G||P|^{-1}|G'|^{-1}|B'|$.