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CHAPTER 1

Representation Theory of Groups - Algebraic Foundations

1.1. Basic definitions, Schur’s Lemma

We assume that the reader is familiar with the fundamental concepts of abstract group theory and linear algebra. A representation of a group $G$ is a homomorphism from $G$ to the group $GL(V)$ of invertible linear operators on $V$, where $V$ is a nonzero complex vector space. We refer to $V$ as the representation space of $\pi$. If $V$ is finite-dimensional, we say that $\pi$ is finite-dimensional, and the degree of $\pi$ is the dimension of $V$. Otherwise, we say that $\pi$ is infinite-dimensional. If $\pi$ is one-dimensional, then $V \cong \mathbb{C}$ and we view $\pi$ as a homomorphism from $G$ to the multiplicative group of nonzero complex numbers. In the above definition, $G$ is not necessarily finite. The notation $(\pi, V)$ will often be used when referring to a representation.

Examples:

(1) If $G$ is a group, we can define a one-dimensional representation of $G$ by $\pi(g) = 1$, $g \in G$. This representation is called the trivial representation of $G$.

(2) Let $G = \mathbb{R}$ and $z \in \mathbb{C}$. The function $t \mapsto e^{zt}$ defines a one-dimensional representation of $G$.

If $n$ is a positive integer and $\mathbb{C}$ is the field of complex numbers, let $GL_n(\mathbb{C})$ denote the group of invertible $n \times n$ matrices with entries in $\mathbb{C}$. If $(\pi, V)$ is a finite-dimensional representation of $G$, then, via a choice of ordered basis $\beta$ for $V$, the operator $\pi(g) \in GL(V)$ is identified with the element $[\pi(g)]_\beta$ of $GL_n(\mathbb{C})$, where $n$ is the degree of $\pi$. Hence we may view a finite-dimensional representation of $G$ as a homomorphism from $G$ to the group $GL_n(\mathbb{C})$.

Examples:

(1) The self-representation of $GL_n(\mathbb{C})$ is the $n$-dimensional representation defined by $\pi(g) = g$.

(2) The function $g \mapsto \det g$ is a one-dimensional representation of $GL_n(\mathbb{C})$.

(3) Let $V$ be a space of functions from $G$ to some complex vector space. Suppose that $V$ has the property that whenever $f \in V$, the function $g_0 \mapsto f(g_0g)$ also belongs to $V$ for all $g \in G$. Then we may define a representation $(\pi, V)$ by $(\pi(g)f)(g_0) = f(g_0g)$, $f \in V$, $g$, $g_0 \in G$. For example, if $G$ is a finite group, we may take $V$ to be the space of all complex-valued functions on $G$. In this case, the resulting representation is called the right regular representation of $G$.

Let $(\pi, V)$ be a representation of $G$. A subspace $W$ of $V$ is stable under the action of $G$, or $G$-invariant, if $\pi(g)w \in W$ for all $g \in G$ and $w \in W$. In this case, denoting the
restriction of $\pi(g)$ to $W$ by $\pi|_{W}(g)$, $(\pi|_{W}, W)$ is a representation of $G$, and we call it a subrepresentation of $\pi$ (or a subrepresentation of $V$).

If $W' \subset W$ are subrepresentations of $\pi$, then each $\pi|_{W}(g)$, $g \in G$, induces an invertible linear operator $\pi|_{W/W'}(g)$ on the quotient space $W/W'$, and $(\pi|_{W/W'}, W/W')$ is a representation of $G$, called a subquotient of $\pi$. In the special case $W = V$, it is called a quotient of $\pi$.

A representation $(\pi, V)$ of $G$ is finitely-generated if there exist finitely many vectors $v_1, \ldots, v_m \in V$ such that $V = \text{Span}\{ \pi(g)v_j \mid 1 \leq j \leq m, \ g \in G \}$. A representation $(\pi, V)$ of $G$ is irreducible if $\{0\}$ and $V$ are the only $G$-invariant subspaces of $V$. If $\pi$ is not irreducible, we say that $\pi$ is reducible.

Suppose that $(\pi_j, V_j)$, $1 \leq j \leq \ell$, are representations of a group $G$. Recall that an element of the direct sum $V = V_1 \oplus \cdots \oplus V_\ell$ can be represented uniquely in the form $v_1 + v_2 + \cdots + v_\ell$, where $v_j \in V_j$. Set

$$\pi(g)(v_1 + \cdots + v_\ell) = \pi_1(g)v_1 + \cdots + \pi_\ell(g)v_\ell, \quad g \in G, \ v_j \in V_j, \ 1 \leq j \leq \ell.$$ 

This defines a representation of $G$, called the direct sum of the representations $\pi_1, \ldots, \pi_\ell$, sometimes denoted by $\pi_1 \oplus \cdots \oplus \pi_\ell$. We may define infinite direct sums similarly. We say that a representation $\pi$ is completely reducible (or semisimple) if $\pi$ is (equivalent to) a direct sum of irreducible representations.

**Lemma.** Suppose that $(\pi, V)$ is a representation of $G$.

1. If $\pi$ is finitely-generated, then $\pi$ has an irreducible quotient.
2. $\pi$ has an irreducible subquotient.

**Proof.** For (1), consider all proper $G$-invariant subspaces $W$ of $V$. This set is nonempty and closed under unions of chains (uses finitely-generated). By Zorn’s Lemma, there is a maximal such $W$. By maximality of $W$, $\pi|_{W}$ is irreducible.

Part (2) follows from part (1) since of $v$ is a nonzero vector in $V$, part (1) says that if $W = \text{Span}\{ \pi(g)v \mid g \in G \}$, then $\pi|_{W}$ has an irreducible quotient. qed

**Lemma.** Let $(\pi, V)$ be a finite-dimensional representation of $G$. Then there exists an irreducible subrepresentation of $\pi$.

**Proof.** If $V$ is reducible, there exists a nonzero $G$-invariant proper subspace $W_1$ of $V$. If $\pi|_{W_1}$ is irreducible, the proof is complete. Otherwise, there exists a nonzero $G$-invariant subspace $W_2$ of $W_1$. Note that $\dim(W_2) < \dim(W_1) < \dim(V)$. Since $\dim(V) < \infty$, this process must eventually stop, that is there exist nonzero subspaces $W_k \subsetneq W_{k-1} \subsetneq \cdots \subsetneq W_1 \subsetneq V$, where $\pi|_{W_k}$ is irreducible. qed

**Lemma.** Let $(\pi, V)$ be a representation of $G$. Assume that there exists an irreducible subrepresentation of $\pi$. The following are equivalent:
(1) \((\pi, V)\) is completely reducible.

(2) For every \(G\)-invariant subspace \(W \subset V\), there exists a \(G\)-invariant subspace \(W'\) such that \(W \oplus W' = V\).

Proof. Assume that \(\pi\) is completely reducible. Without loss of generality, \(\pi\) is reducible. Let \(W\) be a proper nonzero \(G\)-invariant subspace of \(V\). Consider the set of \(G\)-invariant subspaces \(U\) of \(V\) such that \(U \cap W = \{0\}\). This set is nonempty and closed under unions of chains, so Zorn’s Lemma implies existence of a maximal such \(U\). Suppose that \(W \oplus U \neq V\).

Since \(\pi\) is completely reducible, there exists some irreducible subrepresentation \(U'\) such that \(U' \not\subset W \oplus U\). By irreducibility of \(U'\), \(U' \cap (W \oplus U) = \{0\}\). This contradicts maximality of \(U\).

Suppose that (2) holds. Consider the partially ordered set of direct sums of families of irreducible subrepresentations: \(\sum_{\alpha} W_{\alpha} = \oplus_{\alpha} W_{\alpha}\). Zorn’s Lemma applies. Let \(W = \oplus_{\alpha} W_{\alpha}\) be the direct sum for a maximal family. By (2), there exists a subrepresentation \(U\) such that \(V = W \oplus U\). If \(U \neq \{0\}\), according to a lemma above, there exists an irreducible subquotient: \(U \supset U_1 \supset U_2\) such that \(\pi_{U_1/U_2}\) is irreducible. By (2), \(W \oplus U_2\) has a \(G\)-invariant complement \(U_3\): \(V = W \oplus U_2 \oplus U_3\). Now

\[U_3 \simeq V/(W \oplus U_2) = (W \oplus U)/(W \oplus U_2) \simeq U/U_2 \supset U_1/U_2.\]

Identifying \(\pi_{U_1/U_2}\) with an irreducible subrepresentation \(\pi|_{U_4}\) of \(\pi|_{U_3}\), we have \(W \oplus U_4\) contradicting maximality of the family \(W_{\alpha}\). qed

**Lemma.** Subrepresentations and quotient representations of completely reducible representations are completely reducible.

Proof. Let \((\pi, V)\) be a completely reducible representation of \(G\). Suppose that \(W\) is a proper nonzero \(G\)-invariant subspace of \(W\). Then, according to the above lemma, there exists a \(G\)-invariant subspace \(U\) of \(V\) such that \(V = W \oplus U\). It follows that the subrepresentation \(\pi|_W\) is equivalent to the quotient representation \(\pi_{V/U}\). Therefore it suffices to prove that any quotient representation of \(\pi\) is completely reducible.

Let \(\pi_{V/U}\) be an arbitrary quotient representation of \(\pi\). We know that \(\pi = \oplus_{\alpha \in I} \pi_{\alpha}\), where \(I\) is some indexing set, and each \(\pi_{\alpha}\) is irreducible. Let \(pr : V \rightarrow V/U\) be the canonical map. Then \(V/U = pr(V) = \oplus_{\alpha \in I} pr(V_{\alpha})\). Because \(pr(V_{\alpha})\) is isomorphic to a quotient of \(V_{\alpha}\) \((pr(V_{\alpha}) \simeq V_{\alpha}/\ker(pr|V_{\alpha}))\) and \(\pi_{\alpha}\) is irreducible, we have that \(pr(V_{\alpha})\) is either 0 or irreducible. Hence \(\pi_{V/U}\) is completely reducible. qed

**Exercises:**

(1) Show that the self-representation of \(GL_n(\mathbb{C})\) is irreducible.

(2) Verify that \(\pi : t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\) defines a representation of \(\mathbb{R}\), with space \(\mathbb{C}^2\), that is a two-dimensional representation of \(\mathbb{R}\). Show that there is exactly one one-dimensional
subrepresentation, hence $\pi$ is not completely reducible. Prove that the restriction of $\pi$ to the unique one-dimensional invariant subspace $W$ is the trivial representation, and the quotient representation $\pi_{V/W}$ is the trivial representation.

If $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are representations of a group $G$, a linear transformation $A : V_1 \to V_2$ intertwines $\pi_1$ and $\pi_2$ if $A\pi_1(g)v = \pi_2(g)Av$ for all $v \in V_1$ and $g \in G$. The notation $\text{Hom}_G(\pi_1, \pi_2)$ or $\text{Hom}_G(V_1, V_2)$ will be used to denote the set of linear transformations from $V_1$ to $V_2$ that intertwine $\pi_1$ and $\pi_2$. Two representations $(\pi_1, V_1)$ and $(\pi_2, V_2)$ of a group $G$ are said to be equivalent (or isomorphic) whenever $\text{Hom}_G(\pi_1, \pi_2)$ contains an isomorphism, that is, whenever there exists an invertible linear transformation $A : V_1 \to V_2$ that intertwines $\pi_1$ and $\pi_2$. In this case, we write $\pi_1 \simeq \pi_2$. It is easy to check that the notion of equivalence of representations defines an equivalence relation on the set of representations of $G$. It follows from the definitions that if $\pi_1$ and $\pi_2$ are equivalent representations, then $\pi_1$ is irreducible if and only if $\pi_2$ is irreducible. More generally, $\pi_1$ is completely reducible if and only if $\pi_2$ is completely reducible.

**Lemma.** Suppose that $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are finite-dimensional representations of $G$. Then the following are equivalent:

1. $\pi_1$ and $\pi_2$ are equivalent.
2. $\dim V_1 = \dim V_2$ and there exist ordered bases $\beta_1$ and $\beta_2$ of $V_1$ and $V_2$, respectively, such that $[\pi_1(g)]_{\beta_1} = [\pi_2(g)]_{\beta_2}$ for all $g \in G$.

Proof. Assume (1). Fix ordered bases $\gamma_1$ for $V_1$ and $\gamma_2$ for $V_2$. Via these bases, identifying any invertible operator in $\text{Hom}_G(\pi_1, \pi_2)$ as a matrix $A$ in $\text{GL}_n(\mathbb{C})$, we have

$$[\pi_1(g)]_{\gamma_1} = A^{-1}[\pi_2(g)]_{\gamma_2}A, \quad \forall g \in G.$$  

Let $\beta_2 = \gamma_1$. Because $A \in \text{GL}_n(\mathbb{C})$, there exists an ordered basis $\beta_2$ of $V_2$ such that $A$ is the change of basis matrix from $\beta_2$ to $\gamma_2$. With these choices of $\beta_1$ and $\beta_2$, (2) holds.

Now assume that (2) holds. Let $A$ be the unique linear transformation from $V_1$ to $V_2$ which maps the $j$th vector in $\beta_1$ to the $j$th vector in $\beta_2$. qed

A representation $(\pi, V)$ of $G$ has a (finite) composition series if there exist $G$-invariant subspaces $V_j$ of $V$ such that

$$\{0\} \subset V_1 \subset \cdots \subset V_r = V$$

each subquotient $\pi_{V_{j+1}/V_j}$, $1 \leq j \leq r - 1$, is irreducible. The subquotients $\pi_{V_{j+1}/V_j}$ are called the composition factors of $\pi$.

**Lemma.** Let $(\pi, V)$ be a finite-dimensional representation of $G$. Then $\pi$ has a composition series. Up to reordering and equivalence, the composition factors of $\pi$ are unique.

Proof left as an exercise.
Schur’s Lemma. Let \((\pi_1, V_1)\) and \((\pi_2, V_2)\) be irreducible representations of \(G\). Then any nonzero operator in \(\text{Hom}_G(\pi_1, \pi_2)\) is an isomorphism.

Proof. If \(\text{Hom}_G(\pi_1, \pi_2) = \{0\}\) there is nothing to prove, so assume that it is nonzero. Suppose that \(A \in \text{Hom}_G(\pi_1, \pi_2)\) is nonzero. Let \(g \in G\) and \(v_2 \in A(V_1)\). Writing \(v_2 = A(v_1)\), for some \(v_1 \in V_1\), we have \(\pi_2(g)v_2 = A(\pi(g)v_1) \in A(V_1)\). Hence \(A(V_1)\) is a nonzero \(G\)-invariant subspace of \(V_2\). By irreducibility of \(\pi_2\), we have \(A(V_1) = V_2\).

Next, let \(W\) be the kernel of \(A\). Let \(v_1 \in W\). Then \(A(\pi_1(g)v_1) = \pi_2(g)(A(v_1)) = \pi_1(g)0 = 0\) for all \(g \in G\). Hence \(W\) is a \(G\)-invariant proper subspace of \(V_1\). By irreducibility of \(\pi_1\), \(W = \{0\}\). qed

Corollary. Let \((\pi, V)\) be a finite-dimensional irreducible representation of \(G\). Then \(\text{Hom}_G(\pi, \pi)\) consists of scalar multiples of the identity operator, that is, \(\text{Hom}_G(\pi, \pi) \simeq \mathbb{C}\).

Proof. Let \(A \in \text{Hom}_G(\pi, \pi)\). Let \(\lambda \in \mathbb{C}\) be an eigenvalue of \(A\) (such an eigenvalue exists, since \(V\) is finite-dimensional and \(\mathbb{C}\) is algebraically closed). It is easy to see that \(A - \lambda I \in \text{Hom}_G(\pi, \pi)\). But \(A - \lambda I\) is not invertible. By the previous lemma, \(A = \lambda I\). qed

Corollary. If \(G\) is an abelian group, then every irreducible finite-dimensional representation of \(G\) is one-dimensional.

Proof left as an exercise.

Exercise: Prove that an irreducible representation of the cyclic group of order \(n > 1\), with generator \(g_0\), has the form \(g_0^k \mapsto e^{2\pi imk/n}\) for some \(m \in \{0, 1, \ldots, n - 1\}\). (Here \(i\) is a complex number such that \(i^2 = -1\) and \(\pi\) denotes the area of a circle of radius one).

Let \((\pi, V)\) be a representation of \(G\). A matrix coefficient of \(\pi\) is a function from \(G\) to \(\mathbb{C}\) of the form \(g \mapsto \lambda(\pi(g)v)\), for some fixed \(v \in V\) and \(\lambda\) in the dual space \(V^\vee\) of linear functionals on \(V\). Suppose that \(\pi\) is finite-dimensional. Choose an ordered basis \(\beta = \{v_1, \ldots, v_n\}\) of \(V\). Let \(\beta^\vee = \{\lambda_1, \ldots, \lambda_n\}\) be the basis of \(V^\vee\) which is dual to \(\beta\): \(\lambda_j(v_i) = \delta_{ij}, 1 \leq i, j \leq n\). Define a function \(a_{ij} : G \to \mathbb{C}\) by \([\pi(\lambda)v]_\beta = (a_{ij}(\lambda))_{1 \leq i, j \leq n}\). Then it follows from \(\pi(\lambda)v_i = \sum_{\ell=1}^n a_{i\ell}(\lambda)v_\ell\) that \(a_{ij}(\lambda) = \lambda_j(\pi(\lambda)v_i)\), so \(a_{ij}\) is a matrix coefficient of \(\pi\).

If \(g \in G\) and \(\lambda \in V^\vee\), define \(\pi^\vee(\lambda) = \lambda \in V^\vee\) by \((\pi^\vee(\lambda)v)(\nu) = \lambda(\pi(\nu^{-1})v), v \in V\). Then \((\pi^\vee, V^\vee)\) is a representation of \(G\), called the dual (or contragredient) of \(\pi\).

Exercises:

(1) Let \((\pi, V)\) be a finite-dimensional representation of \(G\). Choose \(\beta\) and \(\beta^\vee\) as above. Show that \([\pi^\vee(\lambda)]_{\beta^\vee} = [\pi(\lambda^{-1})]_{\beta}\), for all \(g \in G\). Here the superscript \(t\) denotes transpose.

(2) Prove that if \((\pi, V)\) is finite-dimensional then \(\pi\) is irreducible if and only if \(\pi^\vee\) is irreducible.
(3) Determine whether the self-representation of $GL_n(\mathbb{R})$ (restrict the self-representation of $GL_n(\mathbb{C})$ to the subgroup $GL_n(\mathbb{R})$) is equivalent to its dual.

(4) Prove that a finite-dimensional representation of a finite abelian group is the direct sum of one-dimensional representations.

### 1.2. Tensor products

Let $(\pi_j, V_j)$ be a representation of a group $G_j$, $j = 1, 2$. Recall that $V_1 \otimes V_2$ is spanned by elementary tensors, elements of the form $v_1 \otimes v_2$, $v_1 \in V_1$, $v_2 \in V_2$. We can define a representation $\pi_1 \otimes \pi_2$ of the direct product $G_1 \times G_2$ by setting

$$(\pi_1 \otimes \pi_2)(g_1, g_2)(v_1 \otimes v_2) = \pi_1(g_1)v_1 \otimes \pi_2(g_2)v_2,$$

$g_j \in G_j$, $v_j \in V_j$, $j = 1, 2$,

and extending by linearity to all of $V_1 \otimes V_2$. The representation $\pi_1 \otimes \pi_2$ of $G_1 \times G_2$ is called the \textit{(external or outer) tensor product} of $\pi_1$ and $\pi_2$. Of course, when $\pi_1$ and $\pi_2$ are finite-dimensional, the degree of $\pi_1 \otimes \pi_2$ is equal to the product of the degrees of $\pi_1$ and $\pi_2$.

**Lemma.** Let $(\pi_j, V_j)$ and $G_j$, $j = 1, 2$ be as above. Assume that each $\pi_j$ is finite-dimensional. Then $\pi_1 \otimes \pi_2$ is an irreducible representation of $G_1 \times G_2$ if and only if $\pi_1$ and $\pi_2$ are both irreducible.

**Proof.** If $\pi_1$ or $\pi_2$ is reducible, it is easy to see that $\pi_1 \otimes \pi_2$ is also reducible.

Assume that $\pi_1$ is irreducible. Let $n = \dim V_2$. Let

$$\text{Hom}_{G_1}(\pi_1, \pi_1)^n = \text{Hom}_{G_1}(\pi_1, \pi_1) \oplus \cdots \oplus \text{Hom}_{G_1}(\pi_1, \pi_1),$$

and $\pi_1^n = \pi_1 \oplus \cdots \oplus \pi_1$, where each direct sum has $n$ summands. Then $\text{Hom}_{G_1}(\pi_1, \pi_1)^n \simeq \text{Hom}_{G_1}(\pi_1, \pi_1^n)$, where the isomorphism is given by $A_1 \oplus \cdots \oplus A_n \mapsto B$, with $B(v) = A_1(v) \oplus \cdots \oplus A_n(v)$. By (the corollary to) Schur’s Lemma, $\text{Hom}_{G_1}(\pi_1, \pi_1) \simeq \mathbb{C}$. Irreducibility of $\pi_1$ guarantees that given any nonzero $v \in V_1$, $V_1 = \text{Span}\{ \pi_1(g_1)v \mid g_1 \in G_1 \}$, and this implies surjectivity.

Because $V_2 \simeq \mathbb{C}^n$ and $\mathbb{C} \simeq \text{Hom}_{G_1}(\pi_1, \pi_1)$, we have

$$(i) \quad V_2 \simeq \text{Hom}_{G_1}(\pi_1, \pi_1 \otimes 1^n),$$

where $\pi_1 \otimes 1^n$ is the representation of $G_1$ on $V_1 \otimes V_2$ defined by $(\pi_1 \otimes 1^n)(g_1)(v_1 \otimes v_2) = \pi_1(g_1)v_1 \otimes v_2$, $v_1 \in V_1$, $v_2 \in V_2$. (Note that this representation can be identified with the restriction of $\pi_1 \otimes \pi_2$ to the subgroup $G_1 \times \{1\}$ of $G_1 \times G_2$.

If $m$ is a positive integer, then

$$(ii) \quad V_1 \otimes \text{Hom}_{G_1}(\pi_1, \pi_1^m) \to V_1^m$$

$v \otimes A \mapsto A(v)$
is an isomorphism.

Next, we can use (i) and (ii) to show that

\[ \{ G_1 \text{- invariant subspaces of } V_1 \otimes V_2 \} \leftrightarrow \{ \mathbb{C} \text{- subspaces of } V_2 \} \]

\[ V_1 \otimes W \leftarrow W \]

\[ X \rightarrow \text{Hom}_{G_1}(\pi_1, X) \subset \text{Hom}_{G_1}(\pi_1, \pi_1 \otimes 1^n) = V_2 \]

As any \((G_1 \times G_2)\)-invariant subspace \(X\) of \(V_1 \otimes V_2\) is also a \(G_1\)-invariant subspace, we have \(X = V_1 \otimes W\) for some complex subspace \(W\) of \(V_2\). If \(X \neq \{0\}\) and \(\pi_2\) is irreducible, then

\[ \text{Span}\{ (\pi_1 \otimes \pi_2)(1, g_2)X \mid g_2 \in G_2 \} = V_1 \otimes \text{Span}\{ \pi_2(g_2)W \mid g_2 \in G_2 \} = V_1 \otimes V_2. \]

But \(G_1 \times G_2\)-invariance of \(X\) then forces \(X = V_1 \otimes V_2\). It follows that if \(\pi_1\) and \(\pi_2\) are irreducible, then \(\pi_1 \otimes \pi_2\) is irreducible (as a representation of \(G_1 \times G_2\)). \(\text{qed}\)

**Proposition.** Let \((\pi, V)\) be an irreducible finite-dimensional representation of \(G_1 \times G_2\). Then there exist irreducible representations \(\pi_1\) and \(\pi_2\) of \(G_1\) and \(G_2\), respectively, such that \(\pi \simeq \pi_1 \otimes \pi_2\).

Proof. Note that \(\pi'_1(g_1)v = \pi((g_1, 1))v, g_1 \in G_1, v \in V\), and \(\pi'_2(g_2)v = \pi((1, g_2))v, g_2 \in G_2, v \in V\), define representations of \(G_1\) and \(G_2\), respectively. Choose a nonzero \(G_1\)-invariant subspace \(V_1\) such that \(\pi'_1|_{V_1}\) is an irreducible representation of \(G_1\). Let \(v_0\) be a nonzero vector in \(V_1\). Let

\[ V_2 = \text{Span}\{ \pi'_2(g_2)v_0 \mid g_2 \in G_2 \}. \]

Then \(V_2\) is \(G_2\)-invariant and \(\pi_2 := \pi'_2|_{V_2}\) is a representation of \(G_2\), which might be reducible.

Define \(A : V_1 \otimes V_2 \rightarrow V\) as follows. Let \(v_1 \in V_1\) and \(v_2 \in V_2\). Then there exist complex numbers \(c_j\) and elements \(g_1^{(j)} \in G_1\) such that \(v_1 = \sum_{j=1}^m c_j \pi_1(g_1^{(j)})v_0\), as well as complex numbers \(b_\ell\) and elements \(g_2^{(\ell)} \in G_2\) such that \(v_2 = \sum_{\ell=1}^n b_\ell \pi_2(g_2^{(\ell)})\). Set

\[ A(v_1 \otimes v_2) = \sum_{j=1}^m \sum_{\ell=1}^n c_j b_\ell \pi(g_1^{(j)}, g_2^{(\ell)})v_0. \]

Now \(\pi(g_1^{(j)}, g_2^{(\ell)})v_0 = \pi_1(g_1^{(j)})\pi_2(g_2^{(\ell)})v_0 = \pi_2(g_2^{(\ell)})\pi_1(g_1^{(j)})v_0\). Check that the map \(A\) is well-defined, extending to a linear transformation from \(V_1 \otimes V_2\) to \(V\). Also check that \(A \in \text{Hom}_{G_1 \times G_2}(V_1 \otimes V_2, V)\).

Because \(A(v_0 \otimes v_0) = v_0\), we know that \(A\) is nonzero. Combining \(G_1 \times G_2\)-invariance of \(A(V_1 \otimes V_2)\) with irreducibility of \(\pi\), we have \(A(V_1 \otimes V_2) = V\). If \(A\) also happens to be one-to-one, then we have \(\pi_1 \otimes \pi_2 \simeq \pi\).
Suppose that $A$ is not one-to-one. Then $\text{Ker } A$ is a $G_1 \times G_2$-invariant subspace of $V_1 \otimes V_2$. In particular, $\text{Ker } A$ is a $G_1$-invariant subspace of $V_1 \otimes V_2$. Using irreducibility of $\pi_1$ and arguing as in the previous proof, we can conclude that $\text{Ker } A = V_1 \otimes W$ for some complex subspace $W$ of $V_2$. We have an equivalence of the representations $(\pi_1 \otimes \pi_2)_{(V_1 \otimes V_2)/\text{Ker } A}$ and $\pi$ of $G_1 \times G_2$. To finish the proof, we must show that the quotient representation $(\pi_1 \otimes \pi_2)_{V_1/(V_1 \otimes W)}$ is a tensor product. If $v_1 \in V_1$ and $v_2 \in V_2$, define

$$B(v_1 \otimes (v_1 + W)) = v_1 \otimes v_2 + V_1 \otimes W.$$ 

This extends by linearity to a map from $V_1 \otimes (V_2/W)$ to the quotient space $(V_1 \otimes V_2)/(V_1 \otimes W)$ and it is a simple matter to check that $B$ is an isomorphism and $B \in \text{Hom}_{G_1 \times G_1}(\pi_1 \otimes (\pi_2)_{V_2/W}, (\pi_1 \otimes \pi_2)_{(V_1 \otimes V_2)/(V_1 \otimes W)})$. The details are left as an exercise. qed

If $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are representations of a group $G$, then we may form the tensor product representation $\pi_1 \otimes \pi_2$ of $G \times G$ and restrict to the subgroup $\delta G = \{(g, g) \mid g \in G\}$ of $G \times G$. This restriction is then a representation of $G$, also written $\pi_1 \otimes \pi_2$. It is called the (inner) tensor product of $\pi_1$ and $\pi_2$. Using inner tensor products gives ways to generate new representations of a group $G$. However, it is important to note that even if $\pi_1$ and $\pi_2$ are both irreducible, the inner tensor product representation $\pi_1 \otimes \pi_2$ of $G$ can be reducible.

**Exercise:** Let $\pi_1$ and $\pi_2$ be finite-dimensional irreducible representations of a group $G$. Prove that the trivial representation of $G$ occurs as a subrepresentation of the (inner) tensor product representation $\pi_1 \otimes \pi_2$ of $G$ if and only if $\pi_2$ is equivalent to the dual $\pi_1^\vee$ of $\pi$.

### 1.3. Unitary representations

Suppose that $(\pi, V)$ is a representation of $G$. If $V$ is a finite-dimensional inner product space and there exists an inner product $\langle \cdot, \cdot \rangle$ on $V$ such that

$$\langle \pi(g)v_1, \pi(g)v_2 \rangle = \langle v_1, v_2 \rangle, \quad \forall v_1, v_2 \in V, \ g \in G.$$ 

then we say that $\pi$ is a unitary representation. If $V$ is infinite-dimensional, we say that $\pi$ is pre-unitary if such an inner product exists, and if $V$ is complete with respect to the norm induced by the inner product (that is, $V$ is a Hilbert space), then we say that $\pi$ is unitary.

Now assume that $\pi$ is finite-dimensional. Recall that if $T$ is a linear operator on $V$, the adjoint $T^*$ of $T$ is defined by $\langle T(v), w \rangle = \langle v, T^*(w) \rangle$ for all $v, w \in V$. Note that $\pi$ is unitary if and only if each operator $\pi(g)$ satisfies $\pi(g)^* = \pi(g)^{-1}, \ g \in G$.

Let $n$ be a positive integer. Recall that if $A$ is an $n \times n$ matrix with entries in $\mathbb{C}$, the adjoint $A^*$ of $A$ is just $A^* = {}^t \bar{A}$. 

9
**Lemma.** If \((\pi, V)\) is a finite-dimensional unitary representation of \(G\) and \(\beta\) is an orthonormal basis of \(V\), then \([\pi(g)]_\beta \equiv [\pi(g)]^{-1}_\beta\).

Proof. Results from linear algebra show that if \(T\) is a linear operator on \(V\) and \(\beta\) is an orthonormal basis of \(V\), then \([T^*]_\beta = [T]_\beta^*\). Combining this with \(\pi(g)^* = \pi(g)^{-1}\), \(g \in G\), proves the lemma. \(\mathsf{qed}\)

**Exercises:**

(1) If \((\pi, V)\) is a representation, form a new vector space \(\bar{V}\) as follows. As a set, \(V = \bar{V}\), and \(\bar{V}\) has the same vector addition as \(V\). If \(c \in \mathbb{C}\) and \(v \in \bar{V}\), set \(c \cdot v = \bar{c}v\), where \(\bar{c}\) is the complex conjugate of \(c\) and \(\bar{c}v\) is the scalar multiplication in \(V\). If \(g \in G\), and 
\(v \in \bar{V}\), \(\bar{\pi}(g)v = \pi(g)v\). Show that \((\bar{\pi}, \bar{V})\) is a representation of \(V\).

(2) Assume that \((\pi, V)\) is a finite-dimensional unitary representation. Prove that \(\pi^* \simeq \bar{\pi}\).

**Lemma.** Let \(W\) be a subspace of \(V\), where \((\pi, V)\) is a unitary representation of \(G\). Then \(W\) is \(G\)-invariant if and only if \(W^\perp\) is \(G\)-invariant.

Proof. \(W\) is \(G\)-invariant if and only if \(\pi(g)w \in W\) for all \(g \in G\) and \(w \in W\) if and only if 
\(\langle \pi(g)w, w^\perp \rangle = 0\) for all \(w \in W\), \(w^\perp \in W^\perp\) and \(g \in G\) if and only if 
\(\langle w, \pi(g^{-1})w^\perp \rangle = 0\) for all \(w \in W\), \(w^\perp \in W^\perp\) and \(g \in G\), if and only if \(W^\perp\) is \(G\)-invariant. \(\mathsf{qed}\)

**Corollary.** A finite-dimensional unitary representation is completely reducible.

**Lemma.** Suppose that \((\pi, V)\) is a finite-dimensional unitary representation of \(G\). Let \(W\) be a proper nonzero \(G\)-invariant subspace of \(V\), and let \(P_W\) be the orthogonal projection of \(V\) onto \(W\). Then \(P_W\) commutes with \(\pi(g)\) for all \(g \in G\).

Proof. Let \(w \in W\) and \(w^\perp \in W^\perp\). Then 
\[P_W \pi(g)(w + w^\perp) = P_W \pi(g)w + P_W \pi(g)w^\perp = \pi(g)w + 0 = \pi(g)P_W (w + w^\perp).\]
\(\mathsf{qed}\)

**Lemma.** Let \((\pi, V)\) be a finite-dimensional unitary representation of \(G\). Then \(\pi\) is irreducible if and only if \(\text{Hom}_G(\pi, \pi) \simeq \mathbb{C}\) (every operator which commutes with all \(\pi(g)\)'s is a scalar multiple of the identity operator).

Proof. One direction is simply the corollary to Schur's Lemma (using irreducibility of \(\pi\)). For the other, if \(\pi\) is reducible, and \(W\) is a proper nonzero \(G\)-invariant subspace of \(V\), then \(P_W \in \text{Hom}_G(\pi, \pi)\) and \(P_W\) is not a scalar multiple of the identity operator. \(\mathsf{qed}\)

Suppose that \((\pi_1, V_1)\) and \((\pi_2, V_2)\) are representations of \(G\) and \(V_1\) and \(V_2\) are complex inner product spaces, with inner products \(\langle , \rangle_1\) and \(\langle , \rangle_2\), respectively. Then \(\pi_1\) and \(\pi_2\) are **unitarily equivalent** if there exists an invertible linear operator \(A : V_1 \to V_2\) such that 
\[\langle Av, Aw \rangle_2 = \langle v, w \rangle_1\] for all \(v\) and \(w \in V_1\) and \(A \in \text{Hom}_G(\pi_1, \pi_2)\).
Lemma. Let $(\pi_1, V_1)$ and $(\pi_2, V_2)$ be finite-dimensional unitary representations of $G$. Then $\pi_1 \simeq \pi_2$ if and only if $\pi_1$ and $\pi_2$ are unitarily equivalent.

Proof. Assume that $\pi_1 \simeq \pi_2$. Let $A : V_1 \to V_2$ be an isomorphism such that $A \in \text{Hom}_G(\pi_1, \pi_2)$. Recall that the adjoint $A^* : V_2 \to V_1$ is defined by the condition $\langle A^* v_2, v_1 \rangle_1 = \langle v_2, Av_1 \rangle_2$ for all $v_1 \in V_1$ and $v_2 \in V_2$. By assumption, we have

(i) \[ \pi_1(g) = A^{-1} \pi_2(g) A, \quad \forall \ g \in G. \]

Taking adjoints, we have $\pi_1(g)^* = A^* \pi_2(g)^* (A^*)^{-1}$ for all $g \in G$. Since $\pi_j$ is unitary, we have $\pi_j(g)^* = \pi_j(g^{-1})$. Replacing $g^{-1}$ by $g$, we have

(ii) \[ \pi_1(g) = A^* \pi_2(g) (A^*)^{-1}, \quad \forall \ g \in G. \]

Expressing $\pi_2(g)$ in terms of $\pi_1(g)$ using (i), we can rewrite (ii) as

\[ \pi_1(g) = A^* A \pi_1(g) A^{-1} (A^*)^{-1}, \quad \forall \ g \in G, \]

or

\[ \pi_1(g)^{-1} A^* A \pi_1(g) = A^* A, \quad \forall \ g \in G. \]

Now $A^* A$ is positive definite (that is, self-adjoint and having positive (real) eigenvalues), and so has a unique positive definite square root, say $B$. Note that $\pi_1(g)^{-1} B \pi_1(g)$ is also a square root of $A^* A$ and it is positive definite, using $\pi_1(g)^* = \pi_1(g)^{-1}$. Hence $\pi_1(g)^{-1} B \pi_1(g) = B$ for all $g \in G$. Writing $A$ in terms of the polar decomposition, we have $A = UB$, with $B$ as above, and with $U$ an isomorphism from $V_1 \to V_2$ such that $\langle Uv, Uw \rangle_2 = \langle v, w \rangle_1$ for all $v$ and $w \in V_1$. Next, note that

\[ \pi_2(g) = UB \pi_1(g) B^{-1} U^{-1} = U \pi_1(g) U^{-1}, \quad \forall \ g \in G. \]

Hence $U \in \text{Hom}_G(\pi_1, \pi_2)$, and $\pi_1$ and $\pi_2$ are unitarily equivalent. qed

1.4. Characters of finite-dimensional representations

Let $(\pi, V)$ be a finite-dimensional representation of a group $G$. The function $g \mapsto \text{tr} \pi(g)$ from $G$ to $\mathbb{C}$ is called the character of $\pi$. We use the notation $\chi_\pi(g) = \text{tr} \pi(g)$. Note that we can use any ordered basis of $V$ to compute $\chi_\pi(g)$, since the trace of an operator depends only on the operator itself. Note that if $\pi$ were infinite-dimensional, the operator $\pi(g)$ would not have a trace.
Lemma. Let \((\pi, V)\) be a finite-dimensional representation of \(G\).

(1) If \(\pi' \simeq \pi\), then \(\chi_\pi = \chi_{\pi'}\).

(2) The function \(\chi_\pi\) is constant on conjugacy classes in \(G\).

(3) Let \(\pi^\vee\) be the representation dual to \(\pi\). Then \(\chi_\pi(g) = \chi_{\pi^\vee}(g^{-1}), \ g \in G\).

(4) If \(\pi\) is unitary, then \(\chi_\pi(g^{-1}) = \bar{\chi_\pi(g)}, \ g \in G\).

(5) Suppose that \((\pi, V)\) has a composition series \(\{0\} \subset V_1 \subset \cdots \subset V_r = V\), with composition factors \(\pi_{V_1}, \pi_{V_2/V_1}, \ldots, \pi_{V_r/V_{r-1}}\) (see page 5). Then \(\chi_\pi = \chi_{\pi_{V_1}} + \chi_{\pi_{V_2/V_1}} + \cdots + \chi_{\pi_{V_r/V_{r-1}}}\).

(6) The character \(\chi_{\pi_1 \times \cdots \times \pi_r}\) of a tensor product of finite-dimensional representations \(\pi_1, \ldots, \pi_r\) of \(G_1, \ldots, G_r\), respectively, is given by

\[
\chi_{\pi_1 \times \cdots \times \pi_r}(g_1, \ldots, g_r) = \chi_{\pi_1}(g_1)\chi_{\pi_2}(g_2)\cdots\chi_{\pi_r}(g_r), \quad g_1 \in G_1, \ldots, g_r \in G_r.
\]

Proof. By an earlier result, if \(\pi' \simeq \pi\), then \(\pi'\) and \(\pi\) have the same matrix realization (for some choice of bases). Part (1) follows immediately.

Note that

\[
\chi_{\pi}(g_1g_2^{-1}) = \text{tr}(\pi(g_1)\pi(g)) = \text{tr}(\pi(g)) = \chi_{\pi}(g), \quad g, g_1 \in G.
\]

Recall that if \(\beta\) is an ordered basis of \(V\) and \(\beta^\vee\) is the basis of \(V^\vee\) dual to \(\beta\), then \([\pi(g)]_{\beta} = \tau([\pi^\vee(g^{-1})]_{\beta^\vee}).\) This implies (3).

Suppose that \(\pi\) is unitary. Let \(\beta\) be an orthonormal basis of \(V\). Then \([\pi(g^{-1})]_{\beta} = [\pi(g)]_{\beta} = \tau([\pi(g)]_{\beta^\vee})\) implies part (4).

For (5), it is enough to do the case \(r = 2\). Let \(\beta\) be an ordered basis for \(V_1\). Extend \(\beta\) to an ordered basis \(\gamma\) for \(V_2 = V\). Let \(\gamma\) be the ordered basis for \(V_2/V_1\) which is the image of \(\gamma\) under the canonical map \(V \to V_2/V_1\). Then it is easy to check that \([\pi(g)]_{\gamma}\) is equal to

\[
\begin{pmatrix}
[\pi_{V_1}(g)]_{\beta} & * \\
0 & [\pi_{V_2/V_1}(g)]_{\gamma}
\end{pmatrix}.
\]

For (6), it is enough to do the case \(r = 2\). Let \(\beta = \{v_1, \ldots, v_n\}\) and \(\gamma = \{w_1, \ldots, w_m\}\) be ordered bases of \(V_1\) and \(V_2\), respectively. Then

\[
\{v_j \otimes w_\ell \mid 1 \leq j \leq n, 1 \leq \ell \leq m\}
\]

is an ordered basis of \(V_1 \otimes V_2\). Let \(a_{ij}(g_1)\) be the \(ij\)th entry of \([\pi_1(g_1)]_{\beta}\), \(g_1 \in G_1\), and let \(b_{ij}(g_2)\) be the \(ij\)th entry of \([\pi_2(g_2)]_{\gamma}\), \(g_2 \in G_2\). We have

\[
\pi_1(g_1)v_j = a_{ij}(g_1)v_1 + a_{2j}(g_1)v_2 + \cdots + a_{nj}(g_1)v_n, \quad g_1 \in G_1,
\]

\[
\pi_2(g_2)w_\ell = b_{1\ell}(g_2)w_1 + b_{2\ell}(g_2)w_2 + \cdots + b_{m\ell}(g_2)w_m, \quad g_2 \in G_2.
\]
Hence
\[ \pi_1(g_1)v_j \otimes \pi_2(g_2)w_\ell = \sum_{t=1}^{n} \sum_{s=1}^{m} a_{tj}(g_1)b_{s\ell}(g_2)(v_t \otimes w_s), \]
and, as the coefficient of \( v_j \otimes w_\ell \) on the right side equals \( a_{jj}(g_1)b_{\ell\ell}(g_2) \), we have
\[ \chi_{\pi_1 \otimes \pi_2}(g_1, g_2) = \sum_{j=1}^{n} \sum_{\ell=1}^{m} a_{jj}(g_1)b_{\ell\ell}(g_2) = \chi_{\pi_1}(g_1)\chi_{\pi_2}(g_2), \quad g_1 \in G_1, \ g_2 \in G_2. \]

**Example:** The converse to part (1) is false. Consider the Example (2) on page 4. We have \( \chi_{\pi}(t) = 2 \) for all \( t \in \mathbb{R} \). Now take \( \pi_0 \oplus \pi_0 \), where \( \pi_0 \) is the trivial representation of \( \mathbb{R} \). This clearly has the same character as \( \pi \), though \( \pi_0 \oplus \pi_0 \) is not equivalent to \( \pi \).

In many cases, for example, if \( G \) is finite, or compact, two irreducible finite-dimensional representations having the same character must be equivalent.