MAT 445/1196 - INTRODUCTION TO REPRESENTATION THEORY

CHAPTER 1

Representation Theory of Groups - Algebraic Foundations

1.1. Basic definitions, Schur's Lemma

We assume that the reader is familiar with the fundamental concepts of abstract group theory and linear algebra. A *representation* of a group G is a homomorphism from G to the group $GL(V)$ of invertible linear operators on V, where V is a nonzero complex vector space. We refer to V as the *representation space* of π. If V is finite-dimensional, we say that π is finite-dimensional, and the *degree* of π is the dimension of V. Otherwise, we say that π is infinite-dimensional. If π is one-dimensional, then $V \simeq \mathbb{C}$ and we view π as a homomorphism from G to the multiplicative group of nonzero complex numbers. In the above definition, G is not necessarily finite. The notation (π, V) will often be used when referring to a representation.

Examples:

(1) If G is a group, we can define a one-dimensional representation of G by $\pi(g) = 1$, $g \in G$. This representation is called the trivial representation of G.

(2) Let $G = \mathbb{R}$ and $z \in \mathbb{C}$. The function $t \mapsto e^{zt}$ defines a one-dimensional representation of G.

If n is a positive integer and $\mathbb{C}$ is the field of complex numbers, let $GL_n(\mathbb{C})$ denote the group of invertible $n \times n$ matrices with entries in $\mathbb{C}$. If (π, V) is a finite-dimensional representation of G, then, via a choice of ordered basis β for V, the operator $\pi(g) \in GL(V)$ is identified with the element $[\pi(g)]_\beta$ of $GL_n(\mathbb{C})$, where n is the degree of π. Hence we may view a finite-dimensional representation of G as a homomorphism from G to the group $GL_n(\mathbb{C})$.

Examples:

(1) The *self-representation* of $GL_n(\mathbb{C})$ is the n-dimensional representation defined by $\pi(g) = g$.

(2) The function $g \mapsto \det g$ is a one-dimensional representation of $GL_n(\mathbb{C})$.

(3) Let V be a space of functions from G to some complex vector space. Suppose that V has the property that whenever $f \in V$, the function $g_0 \mapsto f(g_0 g)$ also belongs to V for all $g \in G$. Then we may define a representation (π, V) by $(\pi(g)f)(g_0) = f(gg_0)$, $f \in V$, g, $g_0 \in G$. For example, if G is a finite group, we may take V to be the space of all complex-valued functions on G. In this case, the resulting representation is called the *right regular representation* of G.

Let (π, V) be a representation of G. A subspace W of V is *stable* under the action of G, or *G-invariant*, if $\pi(g)w \in W$ for all $g \in G$ and $w \in W$. In this case, denoting the restriction of $\pi(g)$ to W by $\pi\,|_W(g)$, $(\pi\,|_W, W)$ is a representation of G, and we call it a *subrepresentation* of π (or a *subrepresentation* of V).

If $W' \subset W$ are subrepresentations of π, then each $\pi\,|_W(g)$, $g \in G$, induces an invertible linear operator $\pi_{W/W'}(g)$ on the quotient space W/W', and $(\pi_{W/W'}, W/W')$ is a representation of G, called a *subquotient* of π. In the special case $W = V$, it is called a *quotient* of π.

A representation (π, V) of G is *finitely-generated* if there exist finitely many vectors $v_1, \dots, v_m \in V$ such that $V = \mathrm{Span}\{\, \pi(g)v_j \mid 1 \le j \le m,\ g \in G\,\}$. A representation (π, V) of G is *irreducible* if $\{0\}$ and V are the only G-invariant subspaces of V. If π is not irreducible, we say that π is *reducible*.

Suppose that (π_j, V_j), $1 \le j \le \ell$, are representations of a group G. Recall that an element of the direct sum $V = V_1 \oplus \cdots \oplus V_\ell$ can be represented uniquely in the form $v_1 + v_2 + \cdots + v_\ell$, where $v_j \in V_j$. Set

$$\pi(g)(v_1 + \cdots + v_\ell) = \pi_1(g)v_1 + \cdots + \pi_\ell(g)v_\ell, \qquad g \in G,\ v_j \in V_j,\ 1 \le j \le \ell.$$

This defines a representation of G, called the *direct sum* of the representations $\pi_1, \dots, \pi_\ell$, sometimes denoted by $\pi_1 \oplus \cdots \oplus \pi_\ell$. We may define infinite direct sums similarly. We say that a representation π is *completely reducible* (or *semisimple*) if π is (equivalent to) a direct sum of irreducible representations.

Lemma. *Suppose that (π, V) is a representation of G.*

(1) If π is finitely-generated, then π has an irreducible quotient.

(2) π has an irreducible subquotient.

Proof. For (1), consider all proper G-invariant subspaces W of V. This set is nonempty and closed under unions of chains (uses finitely-generated). By Zorn's Lemma, there is a maximal such W. By maximality of W, $\pi_{V/W}$ is irreducible.

Part (2) follows from part (1) since of v is a nonzero vector in V, part (1) says that if $W = \mathrm{Span}\{\, \pi(g)v \mid g \in G\,\}$, then $\pi\,|_W$ has an irreducible quotient. qed

Lemma. *Let (π, V) be a finite-dimensional representation of G. Then there exists an irreducible subrepresentation of π.*

Proof. If V is reducible, there exists a nonzero G-invariant proper subspace W_1 of V. If $\pi\,|_{W_1}$ is irreducible, the proof is complete. Otherwise, there exists a nonzero G-invariant subspace W_2 of W_1. Note that $\dim(W_2) < \dim(W_1) < \dim(V)$. Since $\dim(V) < \infty$, this process must eventually stop, that is there exist nonzero subspaces $W_k \subsetneq W_{k-1} \subsetneq \cdots \subsetneq W_1 \subsetneq V$, where $\pi\,|_{W_k}$ is irreducible. qed

Lemma. *Let (π, V) be a representation of G. Assume that there exists an irreducible subrepresentation of π. The following are equivalent:*

(1) (π, V) *is completely reducible.*

(2) For every G*-invariant subspace* $W \subset V$*, there exists a* G*-invariant subspace* W' *such that* $W \oplus W' = V$*.*

Proof. Assume that π is completely reducible. Without loss of generality, π is reducible. Let W be a proper nonzero G-invariant subspace of V. Consider the set of G-invariant subspaces U of V such that $U \cap W = \{0\}$. This set is nonempty and closed under unions of chains, so Zorn's Lemma implies existence of a maximal such U. Suppose that $W \oplus U \neq V$. Since π is completely reducible, there exists some irreducible subrepresentation U' such that $U' \not\subset W \oplus U$. By irreduciblity of U', $U' \cap (W \oplus U) = \{0\}$. This contradicts maximality of U.

Suppose that (2) holds. Consider the partially ordered set of direct sums of families of irreducible subrepresentations: $\sum_\alpha W_\alpha = \oplus_\alpha W_\alpha$. Zorn's Lemma applies. Let $W = \oplus_\alpha W_\alpha$ be the direct sum for a maximal family. By (2), there exists a subrepresentation U such that $V = W \oplus U$. If $U \neq \{0\}$, according to a lemma above, there exists an irreducible subquotient: $U \supset U_1 \supset U_2$ such that π_{U_1/U_2} is is irreducible. By (2), $W \oplus U_2$ has a G-invariant complement U_3: $V = W \oplus U_2 \oplus U_3$. Now

$$U_3 \simeq V/(W \oplus U_2) = (W \oplus U)/(W \oplus U_2) \simeq U/U_2 \supset U_1/U_2.$$

Identifying π_{U_1/U_2} with an irreducible subrepresentation $\pi\,|_{U_4}$ of $\pi\,|_{U_3}$, we have $W \oplus U_4$ contradicting maximality of the family W_α. qed

Lemma. *Subrepresentations and quotient representations of completely reducible representations are completely reducible.*

Proof. Let (π, V) be a completely reducible representation of G. Suppose that W is a proper nonzero G-invariant subspace of W. Then, according to the above lemma, there exists a G-invariant subspace U of V such that $V = W \oplus U$. It follows that the subrepresentation $\pi\,|_W$ is equivalent to the quotient representation $\pi_{V/U}$. Therefore it suffices to prove that any quotient representation of π is completely reducible.

Let $\pi_{V/U}$ be an arbitrary quotient representation of π. We know that $\pi = \oplus_{\alpha \in I} \pi_\alpha$, where I is some indexing set, and each π_α is irreducible. Let $pr : V \to V/U$ be the canonical map. Then $V/U = pr(V) = \oplus_{\alpha \in I} pr(V_\alpha)$. Because $pr(V_\alpha)$ is isomorphic to a quotient of V_α ($pr(V_\alpha) \simeq V_\alpha/\ker(pr \mid V_\alpha)$) and π_α is irreducible, we have that $pr(V_\alpha)$ is either 0 or irreducible. Hence $\pi_{V/U}$ is completely reducible. qed

Exercises:

(1) Show that the self-representation of $GL_n(\mathbb{C})$ is irreducible.

(2) Verify that $\pi : t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ defines a representation of $\mathbb{R}$, with space $\mathbb{C}^2$, that is a two-dimensional representation of $\mathbb{R}$. Show that there is exactly one one-dimensional

subrepresentation, hence π is not completely reducible. Prove that the restriction of π to the unique one-dimensional invariant subspace W is the trivial representation, and the quotient representation $\pi_{V/W}$ is the trivial representation.

If (π_1, V_1) and (π_2, V_2) are representations of a group G, a linear transformation $A : V_1 \to V_2$ *intertwines* π_1 and π_2 if $A\pi_1(g)v = \pi_2(g)Av$ for all $v \in V_1$ and $g \in G$. The notation $\mathrm{Hom}_G(\pi_1, \pi_2)$ or $\mathrm{Hom}_G(V_1, V_2)$ will be used to denote the set of linear transformations from V_1 to V_2 that intertwine π_1 and π_2. Two representations (π_1, V_1) and (π_2, V_2) of a group G are said to be *equivalent* (or *isomorphic*) whenever $\mathrm{Hom}_G(\pi_1, \pi_2)$ contains an isomorphism, that is, whenever there exists an invertible linear tranformation $A : V_1 \to V_2$ that intertwines π_1 and π_2. In this case, we write $\pi_1 \simeq \pi_2$. It is easy to check that the notion of equivalence of representations defines an equivalence relation on the set of representations of G. It follows from the definitions that if π_1 and π_2 are equivalent representations, then π_1 is irreducible if and only if π_2 is irreducible. More generally, π_1 is completely reducible if and only if π_2 is completely reducible.

Lemma. *Suppose that (π_1, V_1) and (π_2, V_2) are finite-dimensional representations of G. Then the following are equivalent:*
(1) π_1 and π_2 are equivalent.
(2) $\dim V_1 = \dim V_2$ and there exist ordered bases β_1 and β_2 of V_1 and V_2, respectively, such that $[\pi_1(g)]_{\beta_1} = [\pi_2(g)]_{\beta_2}$ for all $g \in G$.

Proof. Assume (1). Fix ordered bases γ_1 for V_1 and γ_2 for V_2. Via these bases, identifying any invertible operator in $\mathrm{Hom}_G(\pi_1, \pi_2)$ as a matrix A in $GL_n(\mathbb{C})$, we have

$$[\pi_1(g)]_{\gamma_1} = A^{-1}[\pi_2(g)]_{\gamma_2}A, \qquad \forall\, g \in G.$$

Let $\beta_1 = \gamma_1$. Because $A \in GL_n(\mathbb{C})$, there exists an ordered basis β_2 of V_2 such that A is the change of basis matrix from β_2 to γ_2. With these choices of β_1 and β_2, (2) holds.

Now assume that (2) holds. Let A be the unique linear transformation from V_1 to V_2 which maps the jth vector in β_1 to the jth vector in β_2. qed

A representation (π, V) of G has a (finite) *composition series* if there exist G-invariant subspaces V_j of V such that

$$\{0\} \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V$$

each subquotient π_{V_{j+1}/V_j}, $1 \le j \le r-1$, is irreducible. The subquotients π_{V_{j+1}/V_j} are called the *composition factors* of π.

Lemma. *Let (π, V) be a finite-dimensional represntation of G. Then π has a composition series. Up to reordering and equivalence, the composition factors of π are unique.*

Proof left as an exercise.

Schur's Lemma. *Let (π_1, V_1) and (π_2, V_2) be irreducible representations of G. Then any nonzero operator in $\mathrm{Hom}_G(\pi_1, \pi_2)$ is an isomorphism.*

Proof. If $\mathrm{Hom}_G(\pi_1, \pi_2) = \{0\}$ there is nothing to prove, so assume that it is nonzero. Suppose that $A \in \mathrm{Hom}_G(\pi_1, \pi_2)$ is nonzero. Let $g \in G$ and $v_2 \in A(V_1)$. Writing $v_2 = A(v_1)$, for some $v_1 \in V_1$, we have $\pi_2(g)v_2 = A\pi(g)v_1 \in A(V_1)$. Hence $A(V_1)$ is a nonzero G-invariant subspace of V_2. By irreducibility of π_2, we have $A(V_1) = V_2$.

Next, let W be the kernel of A. Let $v_1 \in W$. Then $A(\pi_1(g)v_1) = \pi_2(g)(A(v_1)) = \pi_1(g)0 = 0$ for all $g \in G$. Hence W is a G-invariant proper subspace of V_1. By irreducibility of π_1, $W = \{0\}$. qed

Corollary. *Let (π, V) be a finite-dimensional irreducible representation of G. Then* $\mathrm{Hom}_G(\pi, \pi)$ *consists of scalar multiples of the identity operator, that is,* $\mathrm{Hom}_G(\pi, \pi) \simeq \mathbb{C}$.

Proof. Let $A \in \mathrm{Hom}_G(\pi, \pi)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A (such an eigenvalue exists, since V is finite-dimensional and $\mathbb{C}$ is algebraically closed). It is easy to see that $A - \lambda I \in \mathrm{Hom}_G(\pi, \pi)$. But $A - \lambda I$ is not invertible. By the previous lemma, $A = \lambda I$. qed

Corollary. *If G is an abelian group, then every irreducible finite-dimensional representation of G is one-dimensional.*

Proof left as an exercise.

Exercise: Prove that an irreducible representation of the cyclic group of order $n > 1$, with generator g_0, has the form $g_0^k \mapsto e^{2\pi imk/n}$ for some $m \in \{0, 1, \ldots, n-1\}$. (Here i is a complex number such that $i^2 = -1$ and π denotes the area of a circle of radius one).

Let (π, V) be a representation of G. A *matrix coefficient* of π is a function from G to $\mathbb{C}$ of the form $g \mapsto \lambda(\pi(g)v)$, for some fixed $v \in V$ and λ in the dual space $V^\vee$ of linear functionals on V. Suppose that π is finite-dimensional. Choose an ordered basis $\beta = \{v_1, \ldots, v_n\}$ of V. Let $\beta^\vee = \{\lambda_1, \ldots, \lambda_n\}$ be the basis of $V^\vee$ which is dual to β: $\lambda_j(v_i) = \delta_{ij}$, $1 \le i, j \le n$. Define a function $a_{ij} : G \to \mathbb{C}$ by $[\pi(g)]_\beta = (a_{ij}(g))_{1 \le i,j \le n}$. Then it follows from $\pi(g)v_i = \sum_{\ell=1}^n a_{i\ell}(g)v_\ell$ that $a_{ij}(g) = \lambda_j(\pi(g)v_i)$, so a_{ij} is a matrix coefficient of π.

If $g \in G$ and $\lambda \in V^\vee$, define $\pi^\vee(g)\lambda \in V^\vee$ by $(\pi^\vee(g)\lambda)(v) = \lambda(\pi(g^{-1})v)$, $v \in V$. Then $(\pi^\vee, V^\vee)$ is a representation of G, called the *dual* (or *contragredient*) of π.

Exercises:

(1) Let (π, V) be a finite-dimensional representation of G. Choose β and $\beta^\vee$ as above. Show that $[\pi^\vee(g)]_{\beta^\vee} = [\pi(g^{-1})]_\beta^t$, for all $g \in G$. Here the superscript t denotes transpose.

(2) Prove that if (π, V) is finite-dimensional then π is irreducible if and only if $\pi^\vee$ is irreducible.

(3) Determine whether the self-representation of $GL_n(\mathbb{R})$ (restrict the self-representation of $GL_n(\mathbb{C})$ to the subgroup $GL_n(\mathbb{R})$) is equivalent to its dual.
(4) Prove that a finite-dimensional representation of a finite abelian group is the direct sum of one-dimensional representations.

1.2. Tensor products

Let (π_j, V_j) be a representation of a group G_j, $j = 1, 2$. Recall that $V_1 \otimes V_2$ is spanned by elementary tensors, elements of the form $v_1 \otimes v_2$, $v_1 \in V_1$, $v_2 \in V_2$. We can define a representation $\pi_1 \otimes \pi_2$ of the direct product $G_1 \times G_2$ by setting

$$(\pi_1 \otimes \pi_2)(g_1, g_2)(v_1 \otimes v_2) = \pi_1(g_1)v_1 \otimes \pi_2(g_2)v_2, \qquad g_j \in G_j,\ v_j \in V_j,\ j = 1, 2,$$

and extending by linearity to all of $V_1 \otimes V_2$. The representation $\pi_1 \otimes \pi_2$ of $G_1 \times G_2$ is called the *(external or outer) tensor product* of π_1 and π_2. Of course, when π_1 and π_2 are finite-dimensional, the degree of $\pi_1 \otimes \pi_2$ is equal to the product of the degrees of π_1 and π_2.

Lemma. *Let (π_j, V_j) and G_j, $j = 1, 2$ be as above. Assume that each π_j is finite-dimensional. Then $\pi_1 \otimes \pi_2$ is an irreducible representation of $G_1 \times G_2$ if and only if π_1 and π_2 are both irreducible.*

Proof. If π_1 or π_2 is reducible, it is easy to see that $\pi_1 \otimes \pi_2$ is also reducible.

Assume that π_1 is irreducible. Let $n = \dim V_2$. Let

$$\mathrm{Hom}_{G_1}(\pi_1, \pi_1)^n = \mathrm{Hom}_{G_1}(\pi_1, \pi_1) \oplus \cdots \oplus \mathrm{Hom}_{G_1}(\pi_1, \pi_1),$$

and $\pi_1^n = \pi_1 \oplus \cdots \oplus \pi_1$, where each direct sum has n summands. Then $\mathrm{Hom}_{G_1}(\pi_1, \pi_1)^n \simeq \mathrm{Hom}_{G_1}(\pi_1, \pi_1^n)$, where the isomorphism is given by $A_1 \oplus \cdots \oplus A_n \mapsto B$, with $B(v) = A_1(v) \oplus \cdots \oplus A_n(v)$. By (the corollary to) Schur's Lemma, $\mathrm{Hom}_{G_1}(\pi_1, \pi_1) \simeq \mathbb{C}$. Irreducibility of π_1 guarantees that given any nonzero $v \in V_1$, $V_1 = \mathrm{Span}\{\, \pi_1(g_1)v \mid g_1 \in G_1 \,\}$, and this implies surjectivity.

Because $V_2 \simeq \mathbb{C}^n$ and $\mathbb{C} \simeq \mathrm{Hom}_{G_1}(\pi_1, \pi_1)$, we have

$$(i) \qquad V_2 \simeq \mathrm{Hom}_{G_1}(\pi_1, \pi_1 \otimes 1^n),$$

where $\pi_1 \otimes 1^n$ is the representation of G_1 on $V_1 \otimes V_2$ defined by $(\pi_1 \otimes 1^n)(g_1)(v_1 \otimes v_2) = \pi_1(g_1)v_1 \otimes v_2$, $v_1 \in V_1$, $v_2 \in V_2$. (Note that this representation can be identified with the restriction of $\pi_1 \otimes \pi_2$ to the subgroup $G_1 \times \{1\}$ of $G_1 \times G_2$).

If m is a positive integer, then

$$(ii) \qquad \begin{aligned} V_1 \otimes \mathrm{Hom}_{G_1}(\pi_1, \pi_1^m) &\to V_1^m \\ v \otimes A &\mapsto A(v) \end{aligned}$$

is an isomorphism.

Next, we can use (i) and (ii) to show that

$$\{\, G_1 - \text{invariant subspaces of } V_1 \otimes V_2 \,\} \leftrightarrow \{\, \mathbb{C} - \text{subspaces of } V_2 \,\}$$

$$V_1 \otimes W \leftarrow W$$

$$X \rightarrow \mathrm{Hom}_{G_1}(\pi_1, X) \subset \mathrm{Hom}_{G_1}(\pi_1, \pi_1 \otimes 1^n) = V_2$$

As any $(G_1 \times G_2)$-invariant subspace X of $V_1 \otimes V_2$ is also a G_1-invariant subspace, we have $X = V_1 \otimes W$ for some complex subspace W of V_2. If $X \neq \{0\}$ and π_2 is irreducible, then

$$\mathrm{Span}\{\, (\pi_1 \otimes \pi_2)(1, g_2)X \mid g_2 \in G_2 \,\} = V_1 \otimes \mathrm{Span}\{\, \pi_2(g_2)W \mid g_2 \in G_2 \,\} = V_1 \otimes V_2.$$

But $G_1 \times G_2$-invariance of X then forces $X = V_1 \otimes V_2$. It follows that if π_1 and π_2 are irreducible, then $\pi_1 \otimes \pi_2$ is irreducible (as a representation of $G_1 \times G_2$). qed

Proposition. *Let (π, V) be an irreducible finite-dimensional representation of $G_1 \times G_2$. Then there exist irreducible representations π_1 and π_2 of G_1 and G_2, respectively, such that $\pi \simeq \pi_1 \otimes \pi_2$.*

Proof. Note that $\pi_1'(g_1)v = \pi((g_1, 1))v$, $g_1 \in G_1$, $v \in V$, and $\pi_2'(g_2)v = \pi((1, g_2))v$, $g_2 \in G_2$, $v \in V$, define representations of G_1 and G_2, respectively. Choose a nonzero G_1-invariant subspace V_1 such that $\pi_1' \,|_{V_1}$ is an irreducible representation of G_1. Let v_0 be a nonzero vector in V_1. Let

$$V_2 = \mathrm{Span}\{\, \pi_2'(g_2)v_0 \mid g_2 \in G_2 \,\}.$$

Then V_2 is G_2-invariant and $\pi_2 := \pi_2' \,|_{V_2}$ is a representation of G_2, which might be reducible.

Define $A : V_1 \otimes V_2 \to V$ as follows. Let $v_1 \in V_1$ and $v_2 \in V_2$. Then there exist complex numbers c_j and elements $g_1^{(j)} \in G_1$ such that $v_1 = \sum_{j=1}^m c_j \pi_1(g_1^{(j)})v_0$, as well as complex numbers b_ℓ and elements $g_2^{(\ell)} \in G_2$ such that $v_2 = \sum_{\ell=1}^n b_\ell \pi_2(g_2^{(\ell)})$. Set

$$A(v_1 \otimes v_2) = \sum_{j=1}^m \sum_{\ell=1}^n c_j b_\ell \pi(g_1^{(j)}, g_2^{(\ell)})v_0.$$

Now $\pi(g_1^{(j)}, g_2^{(\ell)})v_0 = \pi_1(g_1^{(j)})\pi_2(g_2^{(\ell)})v_0 = \pi_2(g_2^{(\ell)})\pi_1(g_1^{(j)})v_0$. Check that the map A is well-defined, extending to a linear transformation from $V_1 \otimes V_2$ to V. Also check that $A \in \mathrm{Hom}_{G_1 \times G_2}(V_1 \otimes V_2, V)$.

Because $A(v_0 \otimes v_0) = v_0$, we know that A is nonzero. Combining $G_1 \times G_2$-invariance of $A(V_1 \otimes V_2)$ with irreducibility of π, we have $A(V_1 \otimes V_2) = V$. If A also happens to be one-to-one, then we have $\pi_1 \otimes \pi_2 \simeq \pi$.

Suppose that A is not one-to-one. Then $\operatorname{Ker} A$ is a $G_1 \times G_2$-invariant subspace of $V_1 \otimes V_2$. In particular, $\operatorname{Ker} A$ is a G_1-invariant subspace of $V_1 \otimes V_2$. Using irreducibility of π_1 and arguing as in the previous proof, we can conclude that $\operatorname{Ker} A = V_1 \otimes W$ for some complex subspace W of V_2. We have an equivalence of the representations $(\pi_1 \otimes \pi_2)_{(V_1 \otimes V_2)/\operatorname{Ker} A}$ and π of $G_1 \times G_2$. To finish the proof, we must show that the quotient representation $(\pi_1 \otimes \pi_2)_{V/(V_1 \otimes W)}$ is a tensor product. If $v_1 \in V_1$ and $v_2 \in V_2$, define

$$B(v_1 \otimes (v_1 + W)) = v_1 \otimes v_2 + V_1 \otimes W.$$

This extends by linearity to a map from $V_1 \otimes (V_2/W)$ to the quotient space $(V_1 \otimes V_2)/(V_1 \otimes W)$ and it is a simple matter to check that B is an isomorphism and $B \in \operatorname{Hom}_{G_1 \times G_1}(\pi_1 \otimes (\pi_2)_{V_2/W}, (\pi_1 \otimes \pi_2)_{(V_1 \otimes V_2)/(V_1 \otimes W)}$. The details are left as an exercise. qed

If (π_1, V_1) and (π_2, V_2) are representations of a group G, then we may form the tensor product representation $\pi_1 \otimes \pi_2$ of $G \times G$ and restrict to the subgroup $\delta G = \{\, (g, g) \mid g \in G \,\}$ of $G \times G$. This restriction is then a representation of G, also written $\pi_1 \otimes \pi_2$. It is called the *(inner) tensor product* of π_1 and π_2. Using inner tensor products gives ways to generate new representations of a group G. However, it is important to note that even if π_1 and π_2 are both irreducible, the inner tensor product representation $\pi_1 \otimes \pi_2$ of G can be reducible.

Exercise: Let π_1 and π_2 be finite-dimensional irreducible representations of a group G. Prove that the trivial representation of G occurs as a subrepresentation of the (inner) tensor product representation $\pi_1 \otimes \pi_2$ of G if and only if π_2 is equivalent to the dual $\pi_1^\vee$ of π.

1.3. Unitary representations

Suppose that (π, V) is a representation of G. If V is a finite-dimensional inner product space and there exists an inner product $\langle \cdot, \cdot \rangle$ on V such that

$$\langle \pi(g)v_1, \pi(g)v_2 \rangle = \langle v_1, v_2 \rangle, \qquad \forall\ v_1, v_2 \in V,\ g \in G.$$

then we say that π is a *unitary* representation. If V is infinite-dimensional, we say that π is *pre-unitary* if such an inner product exists, and if V is complete with respect to the norm induced by the inner product (that is, V is a Hilbert space), then we say that π is *unitary.*

Now assume that π is finite-dimensional. Recall that if T is a linear operator on V, the adjoint T^* of T is defined by $< T(v), w >=< v, T^*(w) >$ for all v, $w \in V$. Note that π is unitary if and only if each operator $\pi(g)$ satisfies $\pi(g)^* = \pi(g)^{-1}$, $g \in G$.

Let n be a positive integer. Recall that if A is an $n \times n$ matrix with entries in $\mathbb{C}$, the adjoint A^* of A is just $A^* = {}^t\bar{A}$.

Lemma. *If (π, V) is a finite-dimensional unitary representation of G and β is an orthonormal basis of V, then $[\pi(g)]_\beta^* = [\pi(g)]_\beta^{-1}$.*

Proof. Results from linear algebra show that if T is a linear operator on V and β is an orthonormal basis of V, then $[T^*]_\beta = [T]_\beta^*$. Combining this with $\pi(g)^* = \pi(g)^{-1}$, $g \in G$, proves the lemma. qed

Exercises:

(1) If (π, V) is a representation, form a new vector space $\bar{V}$ as follows. As a set, $V = \bar{V}$, and $\bar{V}$ has the same vector addition as V. If $c \in \mathbb{C}$ and $v \in \bar{V}$, set $c \cdot v = \bar{c}v$, where $\bar{c}$ is the complex conjugate of c and $\bar{c}v$ is the scalar multiplication in V. If $g \in G$, and $v \in \bar{V}$, $\bar{\pi}(g)v = \pi(g)v$. Show that $(\bar{\pi}, \bar{V})$ is a representation of V.

(2) Assume that (π, V) is a finite-dimensional unitary representation. Prove that $\pi^\vee \simeq \bar{\pi}$.

Lemma. *Let W be a subspace of V, where (π, V) is a unitary representation of G. Then W is G-invariant if and only if $W^\perp$ is G-invariant.*

Proof. W is G-invariant if and only if $\pi(g)w \in W$ for all $g \in G$ and $w \in W$ if and only if $\langle \pi(g)w, w^\perp \rangle = 0$ for all $w \in W$, $w^\perp \in W^\perp$ and $g \in G$ if and only if $\langle w, \pi(g^{-1})w^\perp \rangle = 0$ for all $w \in W$, $w^\perp \in W^\perp$ and $g \in G$, if and only if $W^\perp$ is G-invariant. qed

Corollary. *A finite-dimensional unitary representation is completely reducible.*

Lemma. *Suppose that (π, V) is a finite-dimensional unitary represetantion of G. Let W be a proper nonzero G-invariant subspace of V, and let P_W be the orthogonal projection of V onto W. Then P_W commutes with $\pi(g)$ for all $g \in G$.*

Proof. Let $w \in W$ and $w^\perp \in W^\perp$. Then

$$P_W\pi(g)(w + w^\perp) = P_W\pi(g)w + P_W\pi(g)w^\perp = \pi(g)w + 0 = \pi(g)P_W(w + w^\perp).$$

qed

Lemma. *Let (π, V) be a finite-dimensional unitary representation of G. Then π is irreducible if and only if $\mathrm{Hom}_G(\pi, \pi) \simeq \mathbb{C}$ (every operator which commutes with all $\pi(g)$'s is a scalar multiple of the identity operator).*

Proof. One direction is simply the corollary to Schur's Lemma (using irreducibility of π). For the other, if π is reducible, and W is a proper nonzero G-invariant subspace of V, Then $P_W \in \mathrm{Hom}_G(\pi, \pi)$ and P_W is not a scalar multiple of the identity operator. qed

Suppose that (π_1, V_1) and (π_2, V_2) are representations of G and V_1 and V_2 are complex inner product spaces, with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively. Then π_1 and π_2 are *unitarily equivalent* if there exists an invertible linear operator $A : V_1 \to V_2$ such that $\langle Av, Aw \rangle_2 = \langle v, w \rangle_1$ for all v and $w \in V_1$ and $A \in \mathrm{Hom}_G(\pi_1, \pi_2)$.

Lemma. *Let (π_1, V_1) and (π_2, V_2) be finite-dimensional unitary representations of G. Then $\pi_1 \simeq \pi_2$ if and only if π_1 and π_2 are unitarily equivalent.*

Proof. Assume that $\pi_1 \simeq \pi_2$. Let $A : V_1 \to V_2$ be an isomorphism such that $A \in \mathrm{Hom}_G(\pi_1, \pi_2)$. Recall that the adjoint $A^* : V_2 \to V_1$ is defined by the condition $\langle A^* v_2, v_1\rangle_1 = \langle v_2, Av_1\rangle_2$ for all $v_1 \in V_1$ and $v_2 \in V_2$. By assumption, we have

$$(i) \qquad \pi_1(g) = A^{-1}\pi_2(g)A, \quad \forall\, g \in G.$$

Taking adjoints, we have $\pi_1(g)^* = A^*\pi_2(g)^*(A^*)^{-1}$ for all $g \in G$. Since π_j is unitary, we have $\pi_j(g)^* = \pi_j(g^{-1})$. Replacing g^{-1} by g, we have

$$(ii) \qquad \pi_1(g) = A^*\pi_2(g)(A^*)^{-1}, \quad \forall\, g \in G.$$

Expressing $\pi_2(g)$ in terms of $\pi_1(g)$ using (i), we can rewrite (ii) as

$$\pi_1(g) = A^*A\pi_1(g)A^{-1}(A^*)^{-1}, \quad \forall\, g \in G,$$

or

$$\pi_1(g)^{-1}A^*A\pi_1(g) = A^*A, \quad \forall\, g \in G.$$

Now A^*A is positive definite (that is, self-adjoint and having positive (real) eigenvalues), and so has a unique positive definite square root, say B. Note that $\pi_1(g)^{-1}B\pi_1(g)$ is also a square root of A^*A and it is positive definite, using $\pi_1(g)^* = \pi_1(g)^{-1}$. Hence $\pi_1(g)^{-1}B\pi_1(g) = B$ for all $g \in G$. Writing A in terms of the polar decomposition, we have $A = UB$, with B as above, and with U an isomorphism from $V_1 \to V_2$ such that $\langle Uv, Uw\rangle_2 = \langle v, w\rangle_1$ for all v and $w \in V_1$. Next, note that

$$\pi_2(g) = UB\pi_1(g)B^{-1}U^{-1} = U\pi_1(g)U^{-1}, \quad \forall\, g \in G.$$

Hence $U \in \mathrm{Hom}_G(\pi_1, \pi_2)$, and π_1 and π_2 are unitarily equivalent. qed

1.4. Characters of finite-dimensional representations

Let (π, V) be a finite-dimensional representation of a group G. The function $g \mapsto \mathrm{tr}\,\pi(g)$ from G to $\mathbb{C}$ is called the *character* of π. We use the notation $\chi_\pi(g) = \mathrm{tr}\,\pi(g)$. Note that we can use any ordered basis of V to compute $\chi_\pi(g)$, since the trace of an operator depends only on the operator itself. Note that if π were infinite-dimensional, the operator $\pi(g)$ would not have a trace.

Lemma. *Let (π, V) be a finite-dimensional representation of G.*

(1) If $\pi' \simeq \pi$, then $\chi_\pi = \chi_{\pi'}$.

(2) The function χ_π is constant on conjugacy classes in G.

(3) Let $\pi^\vee$ be the representation dual to π. Then $\chi_{\pi^\vee}(g) = \chi_\pi(g^{-1})$, $g \in G$.

(4) If π is unitary, then $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$, $g \in G$.

(5) Suppose that (π, V) has a composition series $\{0\} \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V$, with composition factors $\pi_{V_1}, \pi_{V_2/V_1}, \ldots, \pi_{V_r/V_{r-1}}$ (see page 5). Then $\chi_\pi = \chi_{\pi_{V_1}} + \chi_{\pi_{V_2/V_1}} + \cdots + \chi_{\pi_{V_r/V_{r-1}}}$.

(6) The character $\chi_{\pi_1\otimes\cdots\otimes\pi_r}$ of a tensor product of finite-dimensional representations $\pi_1, \ldots, \pi_r$ of $G_1, \ldots, G_r$, respectively, is given by

$$\chi_{\pi_1\otimes\cdots\otimes\pi_r}(g_1, \ldots, g_r) = \chi_{\pi_1}(g_1)\chi_{\pi_2}(g_2)\cdots\chi_{\pi_r}(g_r), \qquad g_1 \in G_1, \ldots, g_r \in G_r.$$

Proof. By an earlier result, if $\pi' \simeq \pi$, then π' and π have the same matrix realization (for some choice of bases). Part (1) follows immediately.

Note that

$$\chi_\pi(g_1 g g_1^{-1}) = \mathrm{tr}\,(\pi(g_1)\pi(g)\pi(g_1)^{-1}) = \mathrm{tr}\,\pi(g) = \chi_\pi(g), \quad g,\, g_1 \in G.$$

Recall that if β is an ordered basis of V and $\beta^\vee$ is the basis of $V^\vee$ dual to β, then $[\pi(g)]_\beta = {}^t[\pi^\vee(g^{-1})]_{\beta^\vee}$. This implies (3).

Suppose that π is unitary. Let β be an orthonormal basis of V. Then $[\pi(g^{-1})]_\beta = [\pi(g)]_\beta^* = {}^t\overline{[\pi(g)]}$ implies part (4).

For (5), it is enough to do the case $r = 2$. Let β be an ordered basis for V_1. Extend β to an ordered basis γ for $V_2 = V$. Let $\dot\gamma$ be the ordered basis for V_2/V_1 which is the image of γ under the canonical map $V \to V_2/V_1$. Then it is easy to check that $[\pi(g)]_\gamma$ is equal to

$$\begin{pmatrix} [\pi\,|_{V_1}(g)]_\beta & * \\ 0 & [\pi_{V_2/V_1}(g)]_{\dot\gamma} \end{pmatrix}.$$

For (6), it is enough to do the case $r = 2$. Let $\beta = \{\, v_1, \ldots, v_n \}$ and $\gamma = \{\, w_1, \ldots, w_m \,\}$ be ordered bases of V_1 and V_2, respectively. Then

$$\{\, v_j \otimes w_\ell \mid 1 \le j \le n,\ 1 \le \ell \le m \,\}$$

is an ordered basis of $V_1 \otimes V_2$. Let $a_{ij}(g_1)$ be the ijth entry of $[\pi_1(g_1)]_\beta$, $g_1 \in G_1$, and let $b_{ij}(g_2)$ be the ijth entry of $[\pi_2(g_2)]_\gamma$, $g_2 \in G_2$. We have

$$\begin{aligned} \pi_1(g_1)v_j &= a_{1j}(g_1)v_1 + a_{2j}(g_1)v_2 + \cdots + a_{nj}(g_1)v_n, \quad g_1 \in G_r \\ \pi_2(g_2)w_\ell &= b_{1\ell}(g_2)w_1 + b_{2\ell}(g_2)w_2 + \cdots + b_{m\ell}(g_2)w_m, \quad g_2 \in G_2. \end{aligned}$$

Hence

$$\pi_1(g_1)v_j \otimes \pi_2(g_2)w_\ell = \sum_{t=1}^{n}\sum_{s=1}^{m} a_{tj}(g_1)b_{s\ell}(g_2)(v_t \otimes w_s),$$

and, as the coefficient of $v_j \otimes w_\ell$ on the right side equals $a_{jj}(g_1)b_{\ell\ell}(g_2)$, we have

$$\chi_{\pi_1\otimes\pi_2}(g_1, g_2) = \sum_{j=1}^{n}\sum_{\ell=1}^{m} a_{jj}(g_1)b_{\ell\ell}(g_2) = \chi_{\pi_1}(g_1)\chi_{\pi_2}(g_2), \quad g_1 \in G_1,\ g_2 \in G_2.$$

Example: The converse to part (1) is false. Consider the Example (2) on page 4. We have $\chi_\pi(t) = 2$ for all $t \in \mathbb{R}$. Now take $\pi_0 \oplus \pi_0$, where π_0 is the trivial representation of $\mathbb{R}$. This clearly has the same character as π, though $\pi_0 \oplus \pi_0$ is not equivalent to π.

In many cases, for example, if G is finite, or compact, two irreducible finite-dimensional representations having the same character must be equivalent.