Artin's and Brauer's Theorems on Induced Characters

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1 Preliminaries

Let G be a finite group. Every representation of G defines a unique left $\mathbb{C}[G]$-module where $\mathbb{C}[G]$ is the group ring of formal sums of elements of G with coefficients in $\mathbb{C}$; we will refer to representations as left $\mathbb{C}[G]$-modules rather than as homomorphisms from G to $GL(V)$ for some V.

1.1 Definitions of induced representations

Let H be a subgroup of G and W a representation. Then the induced representation $V = i_H^G W$ can be gotten from W by extending the scalar ring from $\mathbb{C}[H]$ to $\mathbb{C}[G]$; in other words, V is

$$\mathbb{C}[G] \otimes_{\mathbb{C}} W \big/ \langle gh \otimes w \sim g \otimes hw, g \in G, h \in H, w \in W \rangle$$

with the action induced linearly from $g(g' \otimes w) = (gg') \otimes w$. We identify each $w \in W$ with $1 \otimes w \in V$. Each $v \in V$ is of the form

$$\sum_{i=1}^{|G|} g_i w_i = \sum_{i=1}^{|G|} |H|^{-1} \sum_{j=1}^{|H|} g_i h_j{}^{-1} h_j w_i,$$

so there is a function $f : G \to W$ such that $f(gh^{-1}) = hf(g)$ and

$$v = \sum_{g \in G} g \otimes f(g);$$

this function is unique, so we can let it be f_v for each v. Define $f_v^{-1}(g) = f_v(g^{-1})$; then $v \mapsto f_v^{-1}$ is an isomorphism from V to $\mathcal{V} = \{f : G \to W | f(hg) = hf(g), g \in G, h \in H\}$ with the action $(gf)(g_0) = f(g_0 g)$; this is the definition of induced representation given in the notes.

1.2 Characters of induced representations

Let χ_W be the character of W. Let R be a system of left coset representatives for H. Then, V as a vector space is $\bigoplus_{r \in R} rW$. Left multiplication by $g \in G$ takes each rW to $r'W$, where $r' \in R$ is the left coset representative for grH. Hence, the trace of the action of g on V is equal to the sum of the traces of the actions of g on each rW such that $gr \in rH$; $grw = r(r^{-1}gr)w$ for each $w \in W$, so this trace is the trace of the action of $r^{-1}gr$ on W. Hence,

$$\chi_V(g) = \sum_{\substack{r \in R \\ r^{-1}gr \in H}} \chi_W(r^{-1}gr).$$

But $(rh)^{-1}grh \in H$ iff $r^{-1}gr \in H$, and then $\chi_W((rh)^{-1}grh) = \chi_W(h^{-1}gh)$, so

$$\chi_V(g) = |H|^{-1} \sum_{\substack{g_0 \in G \\ g_0^{-1} g g_0 \in H}} \chi_W(g_0^{-1} g g_0).$$

2 Artin's Theorem

Let $R(G)$ be the subgroup of $\mathcal{A}(G)$ consisting of all integer linear combinations of characters of representations of G. If V and W are representations of G then $V \otimes_{\mathbb{C}} W$ under the action $g(v \otimes w) = gv \otimes gw$ is another representation, and its character is the product of the characters of V and W. Hence, $R(G)$ is a commutative ring. Then, $\operatorname{Ind}_H^G : \mathcal{A}(H) \to \mathcal{A}(G)$, defined by

$$\operatorname{Ind}_H^G(\chi)(g) = \sum_{\substack{g_0 \in G \\ g_0^{-1} g g_0 \in H}} \chi(g_0^{-1} g g_0),$$

is a homomorphism, and so is $\operatorname{Res}_H^G : \mathcal{A}(G) \to \mathcal{A}(H)$, defined by $\operatorname{Res}_H^G(\chi)(h) = \chi(g)$.

Lemma 1. $\operatorname{Ind}_H^G R(H)$ *is an ideal in* $R(G)$.

Proof. Let $f \in R(H), f_0 \in R(G)$.

$$\begin{aligned}(\operatorname{Ind}_H^G(f) f_0)(g) &= |H|^{-1} \sum_{\substack{g_0 \in G \\ g_0^{-1} g g_0 \in H}} f(g_0^{-1} g g_0) f_0(g) \\ &= |H|^{-1} \sum_{\substack{g_0 \in G \\ g_0^{-1} g g_0 \in H}} f(g_0^{-1} g g_0) f_0(g_0^{-1} g g_0) \\ &= \operatorname{Ind}_H^G(f \operatorname{Res}_H^G(f_0))(g)\end{aligned}$$

Hence the image is closed under multiplication by arbitrary elements of $R(G)$. It is also closed under addition, since $\operatorname{Ind}_H^G(f + f_0) = \operatorname{Ind}_H^G(f) + \operatorname{Ind}_H^G(f_0)$ for all $f, f_0 \in R(H)$. Therefore it's an ideal. □

Let C be the set of cyclic subgroups of G. For each $H \in C$, let $\chi_H(h)$ be

$|H|$ if $H = \langle h \rangle$ and 0 otherwise.

$$\sum_{H \in C} (\mathrm{Ind}_H^G \chi_H)(g) = \sum_{H \in C} \sum_{\substack{g_0 \in G \\ g_0^{-1} g g_0 \in H}} \chi_H(g_0^{-1} g g_0)|H|^{-1} = \sum_{H \in C} \sum_{\substack{g_0 \in G \\ \langle g_0^{-1} g g_0 \rangle = H}} 1 = |G|$$

Lemma 2. *$\chi_H \in R(H)$ for all $H \in C$.*

Proof. We proceed by induction on $|H|$.

$$\chi_H = |H| - \sum_{\substack{K \in C \\ K \subsetneq H}} (\mathrm{Ind}_K^H \chi_K).$$

But the constant function $|H|$ is the character of the trivial representation of degree $|H|$, and $\mathrm{Ind}_K^H \chi_K \in R(H)$ for all $K \in C, |K| < |H|$, so $\chi_H \in R(H)$. □

Theorem 1 (Artin's Theorem). *If V is a representation of G then χ_V is a rational linear combination of characters induced from representations of cyclic subgroups of G.*

Proof. $\sum_{H \in C} \mathrm{Ind}_H^G R(H)$ contains the constant function $|G|$, so since it's an ideal in $R(G)$, it contains $|G|\chi_V$. The conclusion follows. □

3 Brauer's Theorem

Recall that characters of representations of G have values in the ring $A \subset C$ generated by the $|G|$-th roots of unity. Let $AR(G)$ be the ring of linear combinations of characters of G with coefficients in A.

Lemma 3. *Let $\chi : G \to \mathbb{Z}$ be a class function. Then $|G|\chi \in \sum_{H \in C} A \,\mathrm{Ind}_H^G R(H)$.*

Proof.

$$|G|\chi = \sum_{H \in C} (\mathrm{Ind}_H^G \chi_H)\chi = \sum_{H \in C} \mathrm{Ind}_H^G(\chi_H \,\mathrm{Res}_H^G \chi)$$

But for each ψ in the orthonormal basis of $\mathbb{C}R(H)$ consisting of irreducible characters,

$$\left\langle \psi, \chi_H \operatorname{Res}_H^G \chi \right\rangle = \sum_{h\in H} \psi(h)(\chi_H(h^{-1})|H|^{-1})\chi(h^{-1}) \in A,$$

so $\chi_H \operatorname{Res}_H^G \chi \in AR(H)$. The conclusion follows. $\square$

Lemma 4. *Let $\chi \in AR(G)$ have integer values, p be a prime, $g \in G$ have order $p^n l$, where p does not divide l, and $g_0 \in \left\langle g^{p^n} \right\rangle$ so that $gg_0^{-1} \in \left\langle g^l \right\rangle$. Then $\chi(g) \equiv \chi(g_0) \pmod{p}$.*

Proof. Let $H = \langle g \rangle$ and $\chi_0 = \operatorname{Res}_H^G \chi$. Since all irreducible characters of H have degree 1, we can let them be $\psi_i, 1 \le i \le p^n l$, and then $\chi_0 = \sum_{i=1}^{p^n l} a_i\psi_i$ for some $a_i \in A$. $g^{p^n} g_0^{-p^n} \in \left\langle g^{p^n} \right\rangle$ and, since g and g_0 commute, $g^{p^n} g_0^{-p^n} = \left(gg_0^{-1}\right)^{p^n} \in \left\langle g^l \right\rangle \cap \left\langle g^{p^n} \right\rangle = \{1\}$. Hence, $g^{p^n} = g_0^{p^n}$.

$$\begin{aligned}
\chi_0(g)^{p^n} - \chi_0(g_0)^{p^n} &= \left(\sum_{i=1}^{p^n l} a_i\psi_i(g)\right)^{p^n} - \left(\sum_{i=1}^{p^n l} a_i\psi_i(g_0)\right)^{p^n} \\
&\equiv \sum_{i=1}^{p^n l} a_i^{p^n} \psi_i(g)^{p^n} - \sum_{i=1}^{p^n l} a_i^{p^n} \psi_i(g_0)^{p^n} \\
&= \sum_{i=1}^{p^n l} a_i^{p^n} \left(\psi_i\left(g^{p^n}\right) - \psi_i\left(g_0^{p^n}\right)\right) \\
&= 0 \pmod{pA}
\end{aligned}$$

Since χ has integer values, $\chi_0(g)^{p^n} - \chi_0(g_0)^{p^n} \in pA \cap \mathbb{Z} = p\mathbb{Z}$, so $\chi(g) = \chi_0(g) \equiv \chi_0(g)^{p^n} \equiv \chi_0(g_0)^{p^n} \equiv \chi_0(g_0) = \chi(g_0) \pmod{p}$. $\square$

For prime p, let E_p be the set of p-elementary subgroups of G, i.e. subgroups isomorphic to direct products of cyclic groups and p-groups.

Lemma 5. *Let p be a prime and $|G| = p^n l$ where p does not divide l. Then $l \in \sum_{H \in E_p} \operatorname{Ind}_H^G R(H)$.*

Proof. Let R be a system of representatives of the classes in G of elements with order not divisible by p. For each $r \in R$, let $H_r = \langle r \rangle P_r \simeq \langle r \rangle \times P_r$ where P_r is a Sylow p-subgroup of the centralizer $Z(r) = \{g \in G | gr = rg\}$ of r. Let $\chi_r : \langle r \rangle \to \mathbb{C}$ be defined by $\chi_r(g) = \delta_{gr} | \langle r \rangle |$; then $\chi_r \in AR(\langle r \rangle)$ by a lemma proven earlier. Now define $\psi_r : H_r \to \mathbb{C}$ by $\psi_r(gg_0) = \chi_r(g), g \in \langle r \rangle, g_0 \in P_r$; then $\psi_r \in AR(H_r)$, and ψ_r has integer values. Now let $\psi = \sum_{r \in R} \operatorname{Ind}_{H_r}^G \psi_r$. A conjugate of an element with order not divisible by p in H_r for some $r \in R$ must necessarily lie in $\langle r \rangle$. Hence, for each $r, r_0 \in R$,

$$\begin{aligned}
\left(\operatorname{Ind}_{H_r}^G \psi_r\right)(r_0) &= |H_r|^{-1} \sum_{\substack{g \in G \\ g^{-1} r_0 g \in H_r}} \psi_r(g^{-1} r_0 g) \\
&= |H_r|^{-1} \sum_{\substack{g \in G \\ g^{-1} r_0 g \in \langle r \rangle}} \psi_r(g^{-1} r_0 g) \\
&= |P_r|^{-1} \sum_{\substack{g \in G \\ g^{-1} r_0 g = r}} 1 \\
&= \delta_{r_0 r} |P_r|^{-1} \sum_{\substack{g \in G \\ g^{-1} r g = r}} 1 \\
&= \delta_{r_0 r} |P_r|^{-1} |Z(r)|,
\end{aligned}$$

which is divisible by p if and only if $r \neq r_0$. Hence, $\psi(r)$ is not divisible by p for any $r \in R$, so $\psi(g_0)$ is not divisible by p for any g_0 with order not divisible by p. Now given $g \in G$, let $g_0 \in \langle g \rangle$ be such that g_0 has order not divisible by p and gg_0^{-1} has order a power of p. Then, by an earlier lemma, $\psi(g) = \psi(g_0)$. Thus, ψ has integer values not divisible by p, so $l\left(\psi^{p^{n-1}(p-1)} - 1\right)$ has values divisible by $|G|$, so by an earlier lemma $l\left(\psi^{p^{n-1}(p-1)} - 1\right) \in \sum_{H \in E_p} A \operatorname{Ind}_H^G R(H)$. But since $\psi \in \sum_{H \in E_p} A \operatorname{Ind}_H^G R(H)$ and $\sum_{H \in E_p} A \operatorname{Ind}_H^G R(H)$ is an ideal in $AR(G)$, $l\psi^{p^{n-1}(p-1)} \in \sum_{H \in E_p} A \operatorname{Ind}_H^G R(H)$, so $l \in \sum_{H \in E_p} A \operatorname{Ind}_H^G R(H)$. $A \cap \mathbb{Q} = \mathbb{Z}$, so $A/\mathbb{Z}$ is torsion-free, so since it's finitely generated, it's free, so A has a finite basis β with $1 \in \beta$. $l \in R(G)$ can be expressed uniquely as a linear combination of irreducible characters, namely as $l\chi$ where χ is the character of the 1-dimensional trivial representation. But we also know that $l \in \sum_{H \in E_p} A \operatorname{Ind}_H^G R(H)$, so $l \in \sum_{a \in \beta} a \sum_{H \in E_p} \operatorname{Ind}_H^G R(H)$. Hence, $l \in \sum_{H \in E_p} \operatorname{Ind}_H^G R(H)$. □

Theorem 2 (Brauer's Theorem). $R(G) = \sum_{H \in \bigcup_p E_p} \operatorname{Ind}_H^G R(H)$.

Proof. $\sum_{H \in \bigcup_p E_p} \operatorname{Ind}_H^G R(H)$ contains $\sum_{H \in E_p} \operatorname{Ind}_H^G R(H)$ for all primes p, so it has finite index not divisible by p, so it has index 1. The conclusion follows. □

4 Reference

Serre, Jean-Pierre. "Linear Representations of Finite Groups." Springer-Verlag, 1977.