Weyl's Character Formula for Representations of Semisimple Lie Algebras

Ben Reason
University of Toronto

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1 Introduction

Weyl's character formula is a useful tool in understanding the structure of irreducible representations of semisimple Lie algebras. Even more important than the character formula itself is a corollary, the Weyl dimension formula, which gives a fairly simple expression for the total dimension of the representation in terms of the root system of the Lie algebra. Although the results discussed are valid over any algrebraically closed field of characteristic 0, for simplicity all Lie algebras are taken to be complex.

2 Background

2.1 Abstract root systems

Let E be a finite-dimensional real vector space with an inner product. For $\alpha \in E$, the *reflection* through α is the linear map $E \to E$ given by

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \doteq \beta - \langle \beta, \alpha \rangle \alpha$$

.

A *root system* is a subset $\Phi \subset E$, not containing 0, with the following properties:
(i) Φ is finite and spans E
(ii) $\alpha \in E \Rightarrow k\alpha \in E \Leftrightarrow k \in \{-1, 1\}$
(iii) For $\alpha \in \Phi$, $\sigma_\alpha(\Phi) = \Phi$
(iv) For $\alpha, \beta \in \Phi$, $\langle \beta, \alpha \rangle \in \mathbf{Z}$

A subset $\Delta \subset \Phi$ is said to be a *base* if it is a basis for the space E, and every member of Φ is an integer linear combination of elements of Δ that are either all non-negative or all non-positive. Thus, for a fixed base Δ we can decompose Φ into sets (of equal size, by property (ii)) of positive and negative roots, denoted Φ^+ and Φ^- respectively.

The *weights* of root system Φ are the elememts of E satisfying property (iv), ie $\lambda \in E$ is a weight of Φ if $\langle \lambda, \alpha \rangle \in E$ for all $\alpha \in \Phi$. The set of weights is denoted by Λ. A weight λ is said to be *dominant* with respect to a fixed base Δ if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Phi$. The set of dominant weights is denoted Λ^+ (note: this set is only defined wrt some fixed base.)

The subset W of $GL(E)$ generated by the set $\{\sigma_\alpha : \alpha \in \Phi\}$ is called the $Weyl$ group. (We have only defined the action of σ_α on elements of Φ, but property (i) allows us to extend linearly to all of E). In fact, the Weyl group is generated by the reflections through the elements of any fixed base.

After choosing a base Δ, we can define the *length* of $\sigma \in W$, $l(\sigma)$, to the smallest integer n st σ can be written as a product of n reflections through elements in the base. We also define the *sign* of σ by $sn(\sigma) = (-1)^{l(\sigma)}$.

We will need the following facts about the Weyl group:

1. $\langle \sigma\alpha, \sigma\beta \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Phi$ and $\sigma \in W$
2. σ sends $l(\sigma)$ elements of Φ^+ to Φ^- and permutes the rest
3. $|W| < \infty$
4. $l(\sigma) = l(\sigma^{-1})$ for all $\sigma \in W$

2.2 Semisimple Lie algebras

Any semisimple Lie algebra L has a (not necessarily unique) decomposition of the form

$$L = H \bigoplus_{\alpha \in \Phi} L_\alpha$$

where H is a Cartan subalgebra, and each L_α is spanned by an element x_α with the property that $[hx_\alpha] = \alpha(h)\, x_\alpha$ for all $h \in H$, for some $\alpha \in H^*$.

Let $ad : L \to gl(L)$ be the adjoint representation. The $Killing\ form$ of L is a symmetric bilinear form given by

$$\kappa(x, y) = Trace\,(ad\,(x)\, ad\,(y))\,.$$

For $\alpha \in \Phi$, there exists a unique element $t_\alpha \in H$ such that $\alpha(h) = \kappa(t_\alpha, h)$ for all $h \in H$. Under the inner product $(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$, the set $\Phi \subset H^*$ occuring in the decomposition of L becomes a root system.

3 Weyl's Character Formula

For this section, let L, Φ and Δ be fixed.

The irreducible finite-dimensional representations of L are in one-to-one correspondence with the dominant weights of the root system Φ. Let (π, V) be such a representation. Then, V is the direct sum of weight spaces $\{V_\lambda, \lambda \in \Lambda\}$, and V is generated by some *maximal* vector, ie some $v^* \in V$ such that $v^* \in V_\lambda$ and $\pi(x_\alpha) v^* = 0$ for all $\alpha \in \Phi^+$. We identify the representation by the weight of its maximal vector, and write $V = V(\lambda)$.

We define the algebra $Z[\Lambda]$ to be the Z-linear combinations of elements of the form $\{e(\lambda), \lambda \in \Lambda\}$, with multiplication given by $e(\mu) e(\lambda) = e(\lambda + \mu)$. For $\lambda \in \Lambda$, define the *character* of $V(\lambda)$ to be the element of $Z[\Lambda]$ given by

$$Ch_\lambda = \sum_{\mu \in \Lambda} m_\lambda(\mu) e(\mu)$$

where $m_\lambda(\mu)$ denotes the dimension of V_μ in the representation $V(\lambda)$.

Let χ be the set of finitely supported complex-valued linear functionals on H, ie

$$\chi = \{f \in H^* || \{\lambda \in \Lambda | f(\lambda) \neq 0\}| < \infty\}.$$

We can turn χ into an algebra by defining a multiplication operation by *convolution*, ie

$$(fg)(\mu) = f * g(\mu) = \sum_{\nu+\theta=\mu} f(\nu) g(\theta).$$

For $\lambda \in \Lambda$, define $\varepsilon_\lambda \in \chi$ by

$$\varepsilon_\lambda(\mu) = \begin{cases} 1 & \text{if } \mu = \lambda; \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that, under convolution, the function ε_0 is the multiplicative identity.

The map $Z[\Lambda] \to \chi$ which sends each $e(\mu)$ to ε_μ extends to an injective algebra homomorphism, so we can identify any element of $Z[\Lambda]$ with its image in χ. Under this identification, we can finally state Weyl's character formula:

$$\left(\sum_{\sigma \in W} sn(\sigma) \varepsilon_{\sigma\lambda} \right) * Ch_\lambda = \sum_{\sigma \in W} sn(\sigma) \varepsilon_{\sigma(\lambda+\delta)},$$

where

$$\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

In theory, one can find Ch_λ, and hence the dimension of each weight space, by turning Weyl's character formula into a system of linear equations with unknowns $\{m_\lambda(\mu) \,|\mu \in \Lambda\}$. In more complicated examples this calculation becomes tedious, and better algorithms exist for this purpose. However, Weyl's character formula can be used to derive a relatively simple expression for the total dimension of $V(\lambda)$, as seen in example 2 below.

4 Examples

4.1 $L = sl(2, F)$

Let $L = sl(2, F)$, the 2×2 complex matrices with trace zero. If we define

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

we see that $H = span\{h\}$ is a CSA of L, $[hx] = 2x$ and$[hy] = -2y$. Since H is one-dimensional and $\{x, y, h\}$ spans L, there is only one positive root, given by $\alpha(h) = 2$. So $\delta = 1$, and the Weyl group is simply the set $\{1, \sigma\}$, where $\sigma(m) = -m$, for any integer m. Moreover, $\Lambda = Z$, and $\Lambda^+ = Z^+$. For the irreducible representation of highest weight $\lambda \geq 0$, We will use Weyl's character formula to determine the dimension of each weight space.

In this context, the LHS of the character formula reduces to

$$\left(\sum_{\sigma \in W} sn(\sigma) \varepsilon_{\sigma\delta} \right) * ch_\lambda = (\varepsilon_1 - \varepsilon_{-1}) * \left(\sum_{n \in Z} m_\lambda(n) \varepsilon_n \right).$$

The RHS reduces to

$$sn(e) \varepsilon_{e(m+1)} + sn(\sigma) \varepsilon_{\sigma(m+1)} = \varepsilon_{m+1} - \varepsilon_{-m-1}.$$

Evaluating at an integer k we obtain

$$LHS(k) = \sum_{p+q=k} (\varepsilon_1 - \varepsilon_{-1})(p) \left(\sum_{n \in Z} m_\lambda(n) \varepsilon_n(q) \right).$$

The only non-zero terms come when $p = -1, q = k+1$ or $p = 1, q = k-1$, so

$$LHS(k) = \sum_{n \in Z} m_\lambda(n)\left(\varepsilon_n(k-1) - \varepsilon_n(k+1)\right) = m_\lambda(k-1) - m_\lambda(k-1).$$

By equating to the RHS, we see that

$$LHS(k) = \begin{cases} 1 & \text{if } k = \lambda + 1; \\ -1 & \text{if } k = -\lambda - 1; \\ 0 & \text{otherwise.} \end{cases}$$

So, if $k \neq \lambda - 1$ or $\lambda + 1$, then $m_\lambda(k-1) = m_\lambda(k+1)$. Therefore,

$$\ldots = m_\lambda(\lambda - 3) = m_\lambda(\lambda - 1) = m_\lambda(\lambda + 1) = m_\lambda(\lambda + 3) = \ldots,$$

$$m_\lambda(\lambda + 2) = m_\lambda(\lambda + 4) = m_\lambda(\lambda + 6)\ldots,$$

and

$$\ldots = m_\lambda(-\lambda - 6) = m_\lambda(-\lambda - 4) = m_\lambda(-\lambda - 2),$$

so by finite-dimensionality of the representation, all of these are zero. We also have that

$$m_\lambda(-\lambda) = m_\lambda(-\lambda + 2) = \ldots = m_\lambda(\lambda - 2) = m_\lambda(\lambda).$$

Letting $k = \lambda + 1$, we see that $m_\lambda(\lambda) - m_\lambda(\lambda + 2) = 1$, and since we already calculated that $m_\lambda(\lambda + 2) = 0$, we get that $m_\lambda(\lambda) = 1$. This shows that

$$1 = m_\lambda(-\lambda) = m_\lambda(-\lambda + 2) = \ldots = m_\lambda(\lambda - 2) = m_\lambda(\lambda).$$

So,

$$m_\lambda(n) = \begin{cases} 1 & \text{if } n \in \{-\lambda, -\lambda + 2, \ldots, \lambda - 2, \lambda\}; \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the character of $V(\lambda)$ is given by

$$Ch_\lambda = \sum_{n \in Z} m_\lambda(n)\, e(n) = e(-\lambda) + e(-\lambda + 2) + \cdots + e(\lambda - 2) + e(\lambda).$$

4.2 A dimension formula for $V(\lambda)$

Let $V(\lambda)$ be an irreducible representation of a semisimple Lie algebra L. We will use Weyl's character formula to derive a simple formula for the total dimension of $V(\lambda)$. For convenience, define the function $\omega : \Lambda \to \chi$ by

$$\omega(\mu) = \sum_{\sigma \in W} \varepsilon_{\sigma\mu}.$$

So, Weyl's character formula can be written as

$$\omega(\delta) * ch_\lambda = \omega(\lambda + \delta).$$

Since $V(\lambda)$ decomposes into weight spaces, we have that

$$dimV(\lambda) = \sum_{\mu \in \Lambda} m_\lambda(\mu).$$

The entity $\omega(\delta)$ is known as $Weyl's function$, and it can be shown that

$$\omega(\delta) = \varepsilon_{-\delta} * \prod_{\alpha \in \Phi^+} (\varepsilon_\alpha - 1).$$

So, by the identification $e(\mu) \sim \varepsilon_\mu$, we have that $dimV(\lambda)$ is the sum of the values of ch_λ, since the sum of the values of any ε_μ is 1.

Let χ_0 be the subalgebra of χ generated by the functions $\{\varepsilon_\mu, \mu \in \Lambda\}$. Then the map v taking a function f to the sum of its values is a well-defined algebra homomorphism $\chi_0 \to \mathbf{C}$. Note that by the definition of the convolution on χ_0, it is clear that $v(f * g) = v(f)\, v(g)$. What we are looking for is an expression for $v(ch_\lambda)$.

For $\alpha \in \Phi$, the map $\varepsilon_\mu \mapsto (\mu, \alpha)\, \varepsilon_\mu$ extends to an endomorphism η_α of χ_0. Each η_α is in fact a *derivation* of the algebra χ_0, ie

$$\eta_\alpha(f * g) = \eta_\alpha(f) * g + f * \eta_\alpha(g)\, \forall f, g \in \chi_0.$$

The proof of this comes from the fact that the functions $\{\varepsilon_\mu, \mu \in \Lambda\}$ generate the algebra χ_0, combined with the following observation:

$$\eta_\alpha(\varepsilon_\nu * \varepsilon_\mu) = \eta_\alpha(\varepsilon_{\nu+\mu}) = (\nu + \mu, \alpha)\, \varepsilon_{\nu+\mu} = ((\nu, \alpha)\, \varepsilon_\nu) * \varepsilon_\mu + ((\mu, \alpha)\, \varepsilon_\mu) * \varepsilon_\nu$$

$$= \eta_\alpha(\varepsilon_\nu) * \varepsilon_\mu + \varepsilon_\nu * \eta_\alpha(\varepsilon_\mu).$$

We note that the endomorphisms η_β commute with each other, and define

$$\eta = \prod_{\alpha \in \Phi^+} \eta_\alpha \in End(\chi_0).$$

Observe that since each η_β is a derivation,

$$\eta_\beta \prod_{\alpha\in\Phi^+} (\varepsilon_\alpha - 1) = ((\beta,\beta)\,\varepsilon_\beta - 1) * \prod_{\beta\neq\alpha\in\Phi^+} (\varepsilon_\alpha - 1) + (\varepsilon_\beta - 1) * \left(\eta_\beta \left(\prod_{\beta\neq\alpha\in\Phi^+} (\varepsilon_\alpha - 1)\right)\right).$$

Similarly, if

$$\eta^* = \prod_{\alpha\in\Phi'} \eta_\alpha \in End\,(\chi_0)$$

where Φ' is a proper subset of Φ^+, then each term in the expression

$$\eta^* \prod_{\alpha\in\Phi^+} (\varepsilon_\alpha - 1)$$

has some factor of the form $(\varepsilon_\alpha - 1)$.

Keeping these observations in mind, we apply the endomorphism η to both sides of Weyl's character formula to obtain

$$\eta\,(\omega\,(\lambda+\delta)) = \eta\,(\omega\,(\delta) * ch_\lambda) = ch_\lambda * \eta\,(\omega\,(\delta)) + K,$$

where each term in K has a factor of the form $(\varepsilon_\alpha - 1)$. Since $v\,(\varepsilon_\alpha - 1) = v\,(\varepsilon_\alpha) - v\,(\varepsilon_0) = 1 - 1 = 0$, we have that

$$v\circ\eta\,(\omega\,(\lambda+\delta)) = v\circ\eta\,(\omega\,(\delta) * ch_\lambda) = v\,(ch_\lambda * \eta\,(\omega\,(\delta)) + K) = v\circ\eta\,(\omega\,(\delta))\,v\,(ch_\lambda) + 0.$$

This shows that

$$v\,(ch_\lambda) = \frac{v\circ\eta\,(\omega\,(\lambda+\delta))}{v\circ\eta\,(\omega\,(\delta))},$$

so it remains only to find expressions for the numerator and denominator.

Since $\eta_\alpha\,(\varepsilon_\delta) = (\delta,\alpha)\,\varepsilon_\delta$, we get that

$$\eta\,(\varepsilon_\delta) = \prod_{\alpha\in\Phi^+} (\delta,\alpha)\,\varepsilon_\delta$$

$$\Rightarrow v\circ\eta\,(\varepsilon_\delta) = \prod_{\alpha\in\Phi^+} (\delta,\alpha)\,.$$

For $\sigma \in W$, we use the fact that the Weyl group preserves the inner product to derive the similar expression

$$v\circ\eta\,(\varepsilon_{\sigma\delta}) = \prod_{\alpha\in\Phi^+} (\sigma\delta,\alpha) = \prod_{\alpha\in\Phi^+} \left(\delta,\sigma^{-1}\alpha\right)$$

Now, σ^{-1} sends $l(\sigma^{-1}) = l(\sigma)$ positive roots to negative roots and permutes the rest, so we can say

$$\prod_{\alpha\in\Phi^+} \left(\delta, \sigma^{-1}\alpha\right) = sn(\sigma) \prod_{\alpha\in\Phi^+} (\delta, \alpha).$$

Using this, we can evaluate the denominator:

$$v \circ \eta(\omega(\delta)) = \sum_{\sigma\in W} sn(\sigma) v \circ \eta(\varepsilon_{\sigma\delta})$$

$$= \sum_{\sigma\in W} sn(\sigma)^2 \prod_{\alpha\in\Phi^+} (\delta, \alpha) = \sum_{\sigma\in W} \prod_{\alpha\in\Phi^+} (\delta, \alpha) = |W| \prod_{\alpha\in\Phi^+} (\delta, \alpha)$$

Applying the exact same argument to the numerator, we obtain

$$v \circ \eta(\omega(\lambda + \delta)) = |W| \prod_{\alpha\in\Phi^+} (\lambda + \delta, \alpha).$$

Finally, we have

$$dimV(\lambda) = \frac{|W| \prod_{\alpha\in\Phi^+} (\lambda + \delta, \alpha)}{|W| \prod_{\alpha\in\Phi^+} (\delta, \alpha)} = \frac{\prod_{\alpha\in\Phi^+} (\lambda + \delta, \alpha)}{\prod_{\alpha\in\Phi^+} (\delta, \alpha)}.$$

This can also be written as

$$dimV(\lambda) = \frac{\prod_{\alpha\in\Phi^+} \langle \lambda + \delta, \alpha \rangle}{\prod_{\alpha\in\Phi^+} \langle \delta, \alpha \rangle},$$

which we obtain by multiplying the orginal numerator and denominator by

$$\prod_{\alpha\in\Phi^+} \frac{2}{(\alpha, \alpha)}.$$

4.3 Dimension of representations of $O(3, \mathbf{C})$

For a semisimple Lie algebra L with Cartan subalgebra H and corresponding root system Φ, there exists a finite set $\lambda_1, \ldots, \lambda_n$ of dominant weights such that every weight is an integer linear combination of the λ_i's. The dominant weights are precisely those of the form $\lambda = \sum_{i=1 \ldots n} k_i \lambda_i$ where $k_i \in Z^+$. The weights $\{\lambda_1, \ldots, \lambda_n\}$ are referred to as the *fundamental dominant weights*.

Let $0(3, \mathbf{C})$ be the Lie algebra of 3×3 complex matrices x satisfying $xs = -sx^t$, where

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then, taking H to be the subalgebra of diagonal matrices, the corresponding root system has the form $\Delta = \{\alpha, \beta\}, \Phi^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta\}$ where

$$(\alpha, \alpha) = -(\alpha, \beta) = \frac{1}{2}(\beta, \beta) = 1.$$

So, $\delta = 2\alpha + \frac{3}{2}\beta$ and an easy calculation shows that the fundamental dominant weights λ_1, λ_2 are given by $\lambda_1 = \alpha + \frac{1}{2}\beta$, and $\lambda_2 = \alpha + \beta$.

An arbitrary dominant weight is of the form

$$\lambda = m_1\lambda_1 + m_2\lambda_2 = (m_1 + m_2)\,\alpha + \left(\frac{m_1}{2} + m_2\right)\beta$$

where m_1, m_2 are positive integers. So, Weyl's dimension formula says

$$dimV(\lambda) = \frac{\prod_{\gamma\in\Phi^+}(\lambda+\delta, \gamma)}{\prod_{\gamma\in\Phi^+}(\delta, \gamma)}$$

$$= \frac{(\lambda+\delta,\alpha)(\lambda+\delta,\beta)(\lambda+\delta,\alpha+\beta)(\lambda+\delta,2\alpha+\beta)}{(\delta,\alpha)(\delta,\beta)(\delta,\alpha+\beta)(\delta,2\alpha+\beta)}$$

$$= \frac{\left(\frac{m_1+1}{2}\right)(m_2+1)\left(\frac{m_1+3}{2}+m_2\right)(m_1+m_2+2)}{\left(\frac{1}{2}\right)(1)\left(\frac{3}{2}\right)(2)}$$

$$= \frac{1}{6}(m_1+1)(m_2+1)(m_1+2m_2+3)(m_1+m_2+2).$$

5 Works Consulted

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