REPRESENTATION THEORY OF THE SYMMETRIC GROUP: BASIC ELEMENTS

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Since Cayley's theorem implies that every finite group G is isomorphic to a subgroup of $S_{|G|}$, understanding the representation theory of the finite symmetric groups is likely to yield productive tools and results for the representation theory of finite groups in general. In these notes we will examine the basic results for the representation theory of S_n. We will describe all irreducible representations of S_n and their characters, as well as giving a complete method for decomposing a representation of S_n into its irreducible factors, in terms of certain elements of the group algebra $\mathbb{C}S_n$ known as Young symmetrizers. No such general method exists for general finite groups, so in a sense the fundamental problems of representation theory have been answered in the case of finite symmetric groups.

The Group Algebra $\mathbb{C}G$

We first review some facts regarding the group algebra $\mathbb{C}G$ where G is a finite group. If $\mathcal{A}(G)$ is the algebra of all complex valued functions on G with multiplication given by the convolution

$$f * f'(g) = \sum_{g' \in G} f\left(g'\right) f'\left(g'^{-1} g\right),$$

then $\mathbb{C}G$ is isomorphic as a complex algebra with $\mathcal{A}(G)$, via the map $\mathbb{C}G \to \mathcal{A}(G)$, $g \mapsto e_g$, where $e_g : G \to \mathbb{C}$ is the function $g' \mapsto \delta_{g,g'}$. If $a = \sum_{g \in G} a_g g \in \mathbb{C}G$, then a is sent to the function $g \mapsto a_g$. Henceforth we will write $a = \sum_{g \in G} a(g) g$. Then $ab = \sum_{g,g' \in G} a(g) b(g') g g' = \sum_{g \in G} \left(\sum_{g' \in G} a(g') b\left(g'^{-1} g\right) \right) g$, so that as an element in $\mathcal{A}(G)$, $(ab)(g) := a * b(g) = \sum_{g' \in G} a(g') b\left(g'^{-1} g\right)$. In this way we may pass easily between $\mathbb{C}G$ and $\mathcal{A}(G)$. Under this isomorphism we also find that $\mathrm{Z}(\mathbb{C}G)$, the centre of $\mathbb{C}G$, corresponds with $\mathcal{C}(G)$, the set of class functions on G.

On $\mathbb{C}G$ we have the involution

$$a^* = \left(\sum_g a(g) g \right)^* := \sum_g \overline{a(g)} g^{-1} = \sum_g \overline{a\left(g^{-1}\right)} g.$$

Then if $a^* a = 0$ we have $a^* a(g) = \sum_{g' \in G} \overline{a\left(g'^{-1}\right)} a\left(g'^{-1} g\right) = 0$ and in particular, $0 = a^* a(e) = \sum_{g \in G} \overline{a\left(g\right)} a\left(g\right)$, so that $a = 0$. An involution satisfying $a^* a = 0 \Rightarrow a = 0$ is called nondegenerate, and the algebra possessing it is called symmetric. If $a^* = a$ then a is called Hermitian. For a Hermitian element $a \in \mathbb{C}G$, $a^2 = 0 \Rightarrow a = 0$. It follows that if $\mathcal{I}$ is a left ideal of $\mathbb{C}G$ and $\mathcal{I}^2 = \{0\}$ then $\mathcal{I} = 0$, because $a \in \mathcal{I} \Rightarrow (a^* a)^2 \in \mathcal{I}^2$, so that $(a^* a)^2 = 0 \Rightarrow a^* a = 0 \Rightarrow a = 0$.

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If (π, V) is a group representation of the finite group G then it determines the algebra representation $(\hat{\pi}, V)$ of $\mathbb{C}G$, via

$$\hat{\pi}(a) = \sum_{g \in G} a(g)\pi(g) \in \operatorname{End}(V)$$

and it is easily checked that $\hat{\pi}$ is an algebra homomorphism. If (ρ, V) is a representation of $\mathbb{C}G$, then by restriction of ρ to $G \subset \mathbb{C}G$ we obtain a representation $(\tilde{\rho}, V)$ of G, since $\rho(g) \in \mathrm{GL}(V)$ for all $g \in G$. Clearly $\tilde{\hat{\pi}} = \pi$ and $\hat{\tilde{\rho}} = \rho$, and the representation theory of $\mathbb{C}G$ and of G will exhibit corresponding structures. We will denote group representations using π and algebra representations using ρ for clarity.

The left regular representation $(\pi_{\mathrm{L}}, \mathcal{A}(G))$ of G is given by $(\pi_{\mathrm{L}}(g_0) f)(g) = f\left(g_0^{-1} g\right)$, which we may write for $(\pi_{\mathrm{L}}, \mathbb{C}G)$ as $\pi_{\mathrm{L}}(g)(a) = ga$. Then the corresponding left regular representation of the algebra $\mathbb{C}G$ is its self-representation $(\rho_{\mathrm{L}} = \hat{\pi}_{\mathrm{L}}, \mathbb{C}G)$ with $\rho_{\mathrm{L}}(a)(b) = ab$. It follows then that a (vector) subspace $W \subseteq \mathbb{C}G$ is invariant under the left regular representation of $\mathbb{C}G$ if and only if W is a left ideal of $\mathbb{C}G$. A restriction of the left regular representation of $\mathbb{C}G$ to a left ideal W is irreducible if and only if W is a minimal left ideal. Hence the left regular representation of $\mathbb{C}G$ is completely reducible if and only if $\mathbb{C}G = W_1 \oplus \cdots \oplus W_k$, with the W_i minimal left ideals, and this gives the decomposition of the representation into irreducibles. Thus, in order to decompose the left regular representation of $\mathbb{C}G$ into irreducibles it is equivalent to write $\mathbb{C}G$ as a direct sum of minimal left ideals.

In $\mathcal{A}(G)$ we have the inner product

$$(f_1, f_2) = |G|^{-1} \sum_{g \in G} f_1(g)\overline{f_2(g)}$$

which in $\mathbb{C}G$ becomes $\langle a, b\rangle = |G|^{-1} b^* a(e)$. If W is a left ideal in $\mathbb{C}G$ it follows that $W^{\perp}$ is also, and $\mathbb{C}G = W \oplus W^{\perp}$. Then we obtain the following lemma.

Lemma 1. *Every left ideal in $\mathbb{C}G$ is principal and generated by an idempotent.*

Proof. Let W be a left ideal of $\mathbb{C}G$. Since $\mathbb{C}G = W \oplus W^{\perp}$, we write $e = \omega + \omega'$ with $\omega \in W$, $\omega' \in W^{\perp}$. Then $\omega = \omega^2 + \omega\omega' = \omega^2 + \omega'\omega$ and so $\omega\omega' = \omega'\omega \in W \cap W^{\perp} = \{0\}$. It follows that $\omega = \omega^2$ is idempotent, as is $\omega' = (\omega')^2$. Then $(\mathbb{C}G)\,\omega \subseteq W$, $(\mathbb{C}G)\,\omega' \subseteq W^{\perp}$, and if $a \in \mathbb{C}G$ we have $a = a\omega + a\omega'$, so that

$$W \oplus W^{\perp} = \mathbb{C}G = (\mathbb{C}G)\,\omega \oplus (\mathbb{C}G)\,\omega'$$

which shows that $W = (\mathbb{C}G)\,\omega$, $W^{\perp} = (\mathbb{C}G)\,\omega'$. □

In the notation of the lemma, we find also that $W = \{a \in \mathbb{C}G | a\omega = a\}$, $W^{\perp} = \{a \in \mathbb{C}G | a\omega = 0\}$. If $\epsilon \in \mathbb{C}G$ is idempotent, define $\epsilon' = e - \epsilon$, which is also idempotent. We find $\epsilon\epsilon' = \epsilon'\epsilon = 0$ and conclude that $\mathbb{C}G = (\mathbb{C}G)\epsilon \oplus (\mathbb{C}G)\epsilon'$, $(\mathbb{C}G)\epsilon = \{a \in \mathbb{C}G | a\epsilon = a\} = \{a \in \mathbb{C}G | a\epsilon' = 0\}$. However $(\mathbb{C}G)\epsilon$ and $(\mathbb{C}G)\epsilon'$ will not in general be orthogonal.

The search for the irreducible representations of S_n will reduce to the construction of two families of idempotents in $\mathbb{C}S_n$, one of which satisfies an orthogonality property. The first step is more general, showing how the character of a subrepresentation of the left regular representation of G may be expressed in terms of idempotents in $\mathbb{C}G$.

Lemma 2. *If (π, W) is the restriction of the left regular representation $(\pi_{\mathrm{L}}, \mathbb{C}G)$ of G to a left ideal W of $\mathbb{C}G$, i.e. π is a subrepresentation, and if $W = (\mathbb{C}G)\,\epsilon$ for some idempotent ϵ, then we have the following formula for the character of π:*

$$\chi_\pi(g) = \sum_{g' \in G} \epsilon\left(g' g^{-1}\, g'^{\,-1}\right).$$

Proof. Define $P \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}G)$ by $P(a) = a\epsilon$. Define $T : G \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}G)$ by $T(g)(b) = gb\epsilon$. It follows that $T(g) = \pi_{\mathrm{L}}(g) \circ P$, W is $T(g)$-invariant and the restriction of $T(g)$ to W is $\pi(g)$. (I.e. $\pi = T$ as maps $G \to \mathrm{End}_{\mathbb{C}}(W)$.) Let $\epsilon' = e - \epsilon$ and let $W' = (\mathbb{C}G)\epsilon'$. Then $\mathbb{C}G = W \oplus W'$ and P is projection from $\mathbb{C}G$ onto W along W'. $T(g)$ restricts to the zero operator on W'. Choosing bases of W and W' we see that $\operatorname{tr} T(g) = \operatorname{tr} \pi(g) = \chi_\pi(g)$.

Now consider the basis of $\mathbb{C}G$ given by the elements of G in some fixed order: $\{g_1, \ldots, g_n\}$. We have $T(g)\,(g_j) = gg_j\epsilon$, and to discover the i^{th} coordinate in terms of this basis we calculate using the correspondence with $\mathcal{A}(G)$:

$$(gg_j\epsilon)\,(g_i) = \epsilon\left((gg_j)^{-1}\, g_i\right) = \epsilon\left(g_j{}^{-1}\, g^{-1} g_i\right).$$

Hence

$$\chi_\pi(g) = \operatorname{tr} T(g) = \sum_{i=1}^{n} \epsilon\left(g_i{}^{-1}\, g^{-1} g_i\right) = \sum_{g' \in G} \epsilon\left(g' g^{-1}\, g'^{\,-1}\right). \qquad \square$$

For any representation (π, V) of G we may regard V as a $\mathbb{C}G$-module via $a \cdot v := \hat{\pi}(a)(v)$. (This is not in general a faithful module action.) Under this action, idempotents in $\mathbb{C}G$ become projection operators in $\mathrm{End}(V)$.

If $(\pi_1, V_1), \ldots, (\pi_k, V_k)$ is a complete list of irreducible representations of G, then for each $m = 1, \ldots, k$ the projections

$$P_m = (\dim V_m)\, |G|^{-1} \sum_{g \in G} \overline{\chi_{\pi_m}(g)}\pi(g) \in \mathrm{End}(V)$$

give

$$V = \bigoplus_{m=1}^{k} P_m V, \text{ where } P_m V \simeq V_m^{\oplus r_m} \text{ and } \pi\,|_{P_m V} \simeq \pi_m^{\oplus r_m},$$

and so $\pi \simeq \oplus_{m=1}^{k} \pi_m^{\oplus r_m}$, for some nonnegative integer r_m. This gives the decomposition of (π, V) into a direct sum of inequivalent subrepresentations, each of which is equivalent to a multiple (r_m) of some irreducible. In the language of $\mathbb{C}G$ each P_m corresponds to the idempotent

$$\epsilon_m = (\dim V_m)\, |G|^{-1} \sum_{g \in G} \overline{\chi_{\pi_m}(g)} g \in \mathbb{C}G.$$

(The fact that P_m is a projection implies ϵ_m is idempotent follows by calculating P_m for $(\pi, V) = (\pi_{\mathrm{L}}, \mathbb{C}G)$, the left regular representation of G, for then $P_m(e) = \epsilon_m$, and $\epsilon_m^2 = P_m(e)P_m(e) = P_m^2(e) = P_m(e) = \epsilon_m$.) To discover the irreducible factors of (π, V) and not only multiples of them we search for other idempotents of $\mathbb{C}G$, and we see this process carried out for $G = S_n$ below.

A representation (ρ, V) of $\mathbb{C}G$, where $(V, \langle\cdot|\cdot\rangle)$ is a complex inner product space, is called symmetric if $\langle \rho(a)(u) | v\rangle = \langle u | \rho\,(a^*)\,(v)\rangle$ for all $a \in \mathbb{C}G$, $u, v \in V$. If V is finite dimensional, this condition means $\rho\,(a^*) = \rho(a)^*$, the adjoint operator to $\rho(a)$. Thus in an orthonormal basis β for V we have $[\rho\,(a^*)]_\beta = [\rho(a)]_\beta^*$. Then it is easily checked that a representation (π, V) of G is unitary if and only if the representation $(\hat{\pi}, V)$ of $\mathbb{C}G$ is symmetric, where V is finite dimensional.

We use the listed facts freely in what follows below:

- Two representations of a group G are (unitarily) equivalent if and only if the corresponding representations of $\mathbb{C}G$ are (unitarily) equivalent.
- A representation of a group G is irreducible if and only if the corresponding representation of $\mathbb{C}G$ is irreducible.
- $\pi = \pi_1 \oplus \cdots \oplus \pi_k$ as representations of G if and only if $\hat{\pi} = \hat{\pi}_1 \oplus \cdots \oplus \hat{\pi}_k$ as representations of $\mathbb{C}G$.

Representations of S_n

The number of inequivalent finite dimensional representations of S_n is equal to the number of conjugacy classes in S_n. But two elements of S_n are conjugate if and only if they have the same cycle type, i.e. if and only if in their disjoint cycle decompositions they have the same number of k-cycles for each $k = 1, \dots, n$. Since the cycles are disjoint the sum of the number of each type of cycle times its length must be n. Thus the conjugacy classes of S_n are in one to one correspondence with the partitions of n. $\lambda = (\lambda_1, \dots, \lambda_n)$ is a partition of n if $\sum \lambda_i = n$, each λ_i is an integer, and $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. We order the partitions lexicographically: $\lambda > \mu$ if there exists a k such that $\lambda_k > \mu_k$ and $\lambda_i = \mu_i$ for $i < k$.

Each partition λ determines a unique Ferrer's diagram with λ_k dots in the k^{th} row and all rows left-aligned. For example, $\lambda = (3, 2, 2, 1, 1, 0, 0, 0, 0)$ has Ferrer's diagram

$$\begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \\ \bullet & & \\ \bullet & & \end{array} \quad .$$

If we replace each dot by a box we have a Young diagram, and if we write the numbers from 1 to n in the boxes, one per box with no repetitions, in some order, we have a Young tableau. A Young tableau thus corresponds to an element of S_n, namely that element with disjoint cycle decomposition formed by the rows of the tableau, each row mapping to a cycle. (The map from tableaux to S_n is onto but not injective: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$ and $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \\ \cline{1-1} \end{array}$ both produce the transposition $(1\,2)$ in S_3.)

If λ is a partition of n, $A = (a_1, \dots, a_n)$ is a particular ordering of the numbers $1, \dots, n$ and $\tau \in S_n$, define $\tau(A) = (\tau(a_1), \dots, \tau(a_n))$, define Σ_λ to be the Young diagram for λ, define $\Sigma_{\lambda,A}$ to be the Young tableau formed from Σ_λ by filling in the boxes of the diagram with the elements of A, left-to-right, top-to-bottom, and define $\sigma_{\lambda,A} \in S_n$ to be the permutation in S_n produced by $\Sigma_{\lambda,A}$. In case $A = (1, \dots, n)$ we will usually leave off the subscript A, writing Σ_λ instead of $\Sigma_{\lambda,A}$, σ_λ instead of $\sigma_{\lambda,A}$, etc., relying on context to distinguish the different uses. $\Sigma_{\lambda,(1,\dots,n)}$ is considered to be the canonical tableau for the diagram Σ_λ. Finally, define $\tau\Sigma_{\lambda,A} = \Sigma_{\lambda,\tau(A)}$. Then it is easy to verify that $\sigma_{\lambda,\tau(A)} = \tau\sigma_{\lambda,A}\tau^{-1}$.

To obtain a complete set of irreducible representations of S_n we must associate to each partition λ of n an irreducible representation, in such a way that if $\lambda \neq \mu$ are partitions then the associated representations are inequivalent.

We define the so-called Young subgroups of S_n. Given a tableau $\Sigma_{\lambda,A}$ define $P_{\lambda,A}$ to be the subgroup of S_n of permutations sending each row of $\Sigma_{\lambda,A}$ back into itself, and define $Q_{\lambda,A}$ to be the subgroup of S_n of permutations sending each column of $\Sigma_{\lambda,A}$ back into itself. Then $P_{\lambda,A}$ is isomorphic to $S_{\lambda_1} \times \cdots \times S_{\lambda_n}$ (where $S_1 = S_0 = \{e\}$, the trivial group) and $Q_{\lambda,A}$ is isomorphic to $S_{\mu_1} \times \cdots \times S_{\mu_n}$, where μ is the conjugate partition to λ. (The conjugate partition to λ is the partition associated to the diagram which is the transpose of the diagram Σ_λ.) We have $P_{\lambda,\tau(A)} = \tau P_{\lambda,A}\tau^{-1}$ and $Q_{\lambda,\tau(A)} = \tau Q_{\lambda,A}\tau^{-1}$ for all $\tau \in S_n$. Note that $P_{\lambda,A} \cap Q_{\lambda,A} = \{e\}$, from direct application of the definitions.

Now construct the following elements of $\mathbb{C}S_n$:

$$a_{\lambda,A} := \sum_{p\in P_{\lambda,A}} p, \qquad b_{\lambda,A} := \sum_{q\in Q_{\lambda,A}} (\operatorname{sgn} q)q, \qquad c_{\lambda,A} := a_{\lambda,A}b_{\lambda,A}.$$

The following identities hold and are easy to verify:

- $pa_{\lambda,A} = a_{\lambda,A}p = a_{\lambda,A}\ \forall p \in P_{\lambda,A}$
- $(\operatorname{sgn} q)qb_{\lambda,A} = b_{\lambda,A}(\operatorname{sgn} q)q = b_{\lambda,A}\ \forall q \in Q_{\lambda,A}$
- $pc_{\lambda,A}(\operatorname{sgn} q)q = c_{\lambda,A}\ \forall p \in P_{\lambda,A}\ \forall q \in Q_{\lambda,A}$
- $a^2_{\lambda,A} = \lambda! a_{\lambda,A}$ where $\lambda! = (\lambda_1!)\cdots(\lambda_n!) = |P_{\lambda,A}|$
- $b^2_{\lambda,A} = \mu! b_{\lambda,A}$ where μ is the conjugate partition to λ, and $|Q_{\lambda,A}| = \mu!$
- $\sigma a_{\lambda,A}\sigma^{-1} = a_{\lambda,\sigma(A)}\ \forall \sigma \in S_n$
- $\sigma b_{\lambda,A}\sigma^{-1} = b_{\lambda,\sigma(A)}\ \forall \sigma \in S_n$
- $\sigma c_{\lambda,A}\sigma^{-1} = c_{\lambda,\sigma(A)}\ \forall \sigma \in S_n$
- $a^*_{\lambda,A} = a_{\lambda,A}$
- $b^*_{\lambda,A} = b_{\lambda,A}$

Lemma 3. *If $\lambda$and μ are partitions of n with $\lambda \geq \mu$, and no two elements in the same column of the tableau $\Sigma_{\mu,B}$ are in the same row of the tableau $\Sigma_{\lambda,A}$, then (a) $\lambda = \mu$ and (b) $\Sigma_{\mu,B} = pq\Sigma_{\lambda,A}$ for some $p \in P_{\lambda,A}$, $q \in Q_{\lambda,A}$.*

Proof. The first row of $\Sigma_{\lambda,A}$ has λ_1 elements, which by hypothesis occur in λ_1 distinct columns of $\Sigma_{\mu,B}$. Hence $\mu_1 \geq \lambda_1$, and so we must have $\lambda_1 = \mu_1$. Then for some $q_1 \in Q_{\mu,B}$ and $p_1 \in P_{\lambda,A}$ the tableaux $\Sigma_{\lambda,p_1(A)} = p_1\Sigma_{\lambda,A}$ and $\Sigma_{\mu,q_1(B)} = q_1\Sigma_{\mu,B}$ have the same first row. The new tableaux continue to satisfy the hypotheses in the statement of the lemma.

Suppose that for some $k \leq n$ we have shown that $\lambda_1 = \mu_1, \ldots, \lambda_{k-1} = \mu_{k-1}$, and that for some $p \in P_{\lambda,A}$ and $q \in Q_{\mu,B}$ the tableaux $\Sigma_{\lambda,p(A)}$, $\Sigma_{\mu,q(B)}$ satisfy the hypotheses of the lemma and have the same top $k-1$ rows. If $\lambda_k = 0$ then $\mu_k = 0$, and so $\lambda = \mu$ and $\Sigma_{\lambda,p(A)} = \Sigma_{\mu,q(B)}$. Otherwise $\lambda_k > 0$ and the k^{th} row of $\Sigma_{\lambda,p(A)}$ has λ_k elements which appear in distinct columns of $\Sigma_{\mu,q(B)}$, and necessarily all occur in rows k to n of $\Sigma_{\mu,q(B)}$. Hence $\mu_k \geq \lambda_k$, which forces $\lambda_k = \mu_k$. Then for some $q_k \in Q_{\mu,q(B)}$ and $p_k \in P_{\lambda,p(A)}$, $p_k\Sigma_{\lambda,p(A)}$ and $q_k\Sigma_{\mu,q(B)}$ have identical top k rows. (Clearly we may choose q_k so that it leaves unchanged the first $k-1$ rows of $\Sigma_{\mu,q(B)}$ and we may choose p_k so that it leaves unchanged all rows of $\Sigma_{\lambda,p(A)}$ except the k^{th}.)

We find $Q_{\mu,q(B)} = qQ_{\mu,B}q^{-1} = Q_{\mu,B}$ since $q \in Q_{\mu,B}$ and similarly $P_{\lambda,p(A)} = P_{\lambda,A}$. It follows that $q_kq \in Q_{\mu,B}$ and $p_kp \in P_{\lambda,A}$, and the new tableaux $\Sigma_{\lambda,p_kp(A)}$, $\Sigma_{\mu,q_kq(B)}$ satisfy our induction hypothesis. Therefore $\lambda = \mu$ and for some $p \in P_{\lambda,A}$, $q \in Q_{\mu,B}$ we have $\Sigma_{\lambda,p(A)} = \Sigma_{\mu,q(B)}$. Thus we've shown that $\Sigma_{\mu,B} = q^{-1}p\Sigma_{\lambda,A}$. But notice that $pQ_{\lambda,A}p^{-1} = Q_{\lambda,p(A)} = Q_{\mu,q(B)} = qQ_{\mu,B}q^{-1} = Q_{\mu,B}$. Hence $q = pq'p^{-1}$ for some $q' \in Q_{\lambda,A}$, and we have $\Sigma_{\mu,B} = p\,{q'}^{-1}\,\Sigma_{\lambda,A}$ as required. □

This combinatorial lemma together with the identities above have the following consequences.

Corollary 1.

(1) *If* $\lambda \neq \mu$ *then* $0 = a_{\lambda,A} x b_{\mu,B} = b_{\lambda,A} x a_{\mu,B}$ $\forall x \in \mathbb{C}S_n$.
(2) *If* $\lambda \neq \mu$ *then* $c_{\lambda,A} x c_{\mu,B} = 0$ $\forall x \in \mathbb{C}S_n$.
(3) *If* $\forall p \in P_{\lambda,A}$ $\forall q \in Q_{\lambda,A}$ $pa(\operatorname{sgn} q)q = a$, *then* $a = a(e)c_{\lambda,A}$.
(4) $c_{\lambda,A} x c_{\lambda,B} = m\sigma c_{\lambda,B}$ *for some* $m \in \mathbb{C}$, *where* $\sigma \in S_n$ *satisfies* $\sigma(B) = A$. *We have* $m = c_{\lambda,B}\sigma^{-1} x c_{\lambda,B}(e)$.

Proof.

(1) Suppose $\lambda > \mu$. Then by Lemma 3 there exist distinct $i, j \in 1, \ldots, n$ in some column of $\Sigma_{\mu,B}$ such that i and j occur in the same row of $\Sigma_{\lambda,A}$. Let $\tau = (i\ j)$ be their transposition. Then $\tau \in P_{\lambda,A} \cap Q_{\mu,B}$ and hence $a_{\lambda,A} b_{\mu,B} = a_{\lambda,A}\tau(\operatorname{sgn}\tau)\tau b_{\mu,B} = -a_{\lambda,A} b_{\mu,B} \Rightarrow a_{\lambda,A} b_{\mu,B} = 0$. Now let $\sigma \in S_n$. Then $a_{\lambda,A}\sigma b_{\mu,B}\sigma^{-1} = a_{\lambda,A} b_{\mu,\sigma(B)} = 0 \Rightarrow a_{\lambda,A}\sigma b_{\mu,B} = 0$, and hence for all $x \in \mathbb{C}S_n$, $a_{\lambda,A} x b_{\mu,B} = 0$. A symmetric argument shows $b_{\lambda,A} x a_{\mu,B} = 0$ also.

Next suppose $\lambda < \mu$ and let $x \in \mathbb{C}S_n$. Then

$$
\begin{aligned}
(a_{\lambda,A} x b_{\mu,B})^* &= b^*_{\mu,B} x^* a^*_{\lambda,A} && (ab)^* = b^*a^* \text{ in } \mathbb{C}G \text{ can be verified by direct computation using the convolution formula} \\
&= b_{\mu,B} x^* a_{\lambda,A} && \text{since } a_{\lambda,A},\ b_{\mu,B} \text{ are Hermitian} \\
&= 0 && \text{since } \mu > \lambda
\end{aligned}
$$

which implies that $a_{\lambda,A} x b_{\mu,B} = 0$. A symmetric argument then shows that $b_{\lambda,A} x a_{\mu,B} = 0$ also.

(2) $c_{\lambda,A} x c_{\mu,B} = a_{\lambda,A}\left(b_{\lambda,A} x a_{\mu,B}\right) b_{\mu,B} = 0$, by (1).

(3) We have $\forall p \in P_{\lambda,A}$ $\forall q \in Q_{\lambda,A}$,

$$
\begin{aligned}
\sum_{\sigma \in S_n} a(\sigma)\sigma &= \sum_{\sigma \in S_n} (\operatorname{sgn} q) a(\sigma) p\sigma q \\
&= \sum_{\sigma \in S_n} (\operatorname{sgn} q) a\left(p^{-1}\sigma q^{-1}\right)\sigma \\
\Rightarrow a(\sigma) &= (\operatorname{sgn} q) a\left(p^{-1}\sigma q^{-1}\right) \qquad \forall \sigma \in S_n \\
\Rightarrow a\left(p\sigma q\right) &= (\operatorname{sgn} q) a\left(\sigma\right) \qquad \forall \sigma \in S_n. \qquad (*)
\end{aligned}
$$

Hence $a(pq) = (\operatorname{sgn} q)a(e)$. If we can show that $a(\sigma) = 0$ whenever $\sigma \notin P_{\lambda,A}Q_{\lambda,A} := \{pq | p \in P_{\lambda,A}, q \in Q_{\lambda,A}\}$, then

$$
a = \sum_{pq \in P_{\lambda,A}Q_{\lambda,A}} a(pq)pq = a(e) \sum_{\substack{p \in P_{\lambda,A} \\ q \in Q_{\lambda,A}}} (\operatorname{sgn} q) pq = a(e)c_{\lambda,A}
$$

since $P_{\lambda,A} \cap Q_{\lambda,A} = \{e\}$ implies that the set $P_{\lambda,A}Q_{\lambda,A}$ is in bijection with $P_{\lambda,A} \times Q_{\lambda,A}$. Suppose that $\sigma \notin P_{\lambda,A}Q_{\lambda,A}$. Then $\Sigma_{\lambda,\sigma(A)} = \sigma\Sigma_{\lambda,A}$ and so by Lemma 3 there exist i, j in some column of $\Sigma_{\lambda,\sigma(A)}$ such that both lie in the same row of $\Sigma_{\lambda,A}$. Let $\tau = (i\ j)$, so $\tau \in P_{\lambda,A} \cap Q_{\lambda,\sigma(A)}$. Since

$Q_{\lambda,\sigma(A)} = \sigma Q_{\lambda,A}\sigma^{-1}$, $\tau = \sigma q \sigma^{-1}$ for some $q \in Q_{\lambda,A}$. Then $e = \tau^2 = \tau\sigma q \sigma^{-1} \Rightarrow \tau\sigma q = \sigma$. Applying equation $(*)$ we find

$$a(\sigma) = a(\tau\sigma q) = (\operatorname{sgn} q)a(\sigma) = (\operatorname{sgn}\tau)a(\sigma) = -a(\sigma)$$

whence $a(\sigma) = 0$.

(4) Let $x \in \mathbb{C}S_n$, $p \in P_{\lambda,A}$, $q \in Q_{\lambda,A}$. Then $pc_{\lambda,A}xc_{\lambda,A}(\operatorname{sgn} q)q = c_{\lambda,A}xc_{\lambda,A}$ since $c_{\lambda,A} = a_{\lambda,A}b_{\lambda,A}$. Hence $c_{\lambda,A}xc_{\lambda,A} = c_{\lambda,A}xc_{\lambda,A}(e)c_{\lambda,A}$, by (3). Now, if $\Sigma_{\lambda,A}$, $\Sigma_{\lambda,B}$ are two tableaux, there is a $\sigma \in S_n$ such that $A = \sigma(B)$ and $\Sigma_{\lambda,A} = \Sigma_{\lambda,\sigma(B)}$. Then

$$\begin{aligned} c_{\lambda,A}xc_{\lambda,B} &= \sigma c_{\lambda,B}\sigma^{-1}xc_{\lambda,B} \\ &= \left(c_{\lambda,B}\sigma^{-1}xc_{\lambda,B}\right)(e)\sigma c_{\lambda,B}. \end{aligned}$$

□

Corollary 2. *$c_{\lambda,A}c_{\mu,B} = 0$ if $\lambda \neq \mu$. $c^2_{\lambda,A} = c^2_{\lambda,A}(e)c_{\lambda,A}$. $c^2_{\lambda,A}(e) = c^2_\lambda(e)$, i.e. this value depends only on the diagram (so, only on the partition λ), not on the particular tableau chosen. (Recall that $c_\lambda := c_{\lambda,(1,\ldots,n)}$.)*

Proof. The first two statements follow immediately from Corollary 1. For the third statement, we have $A = \sigma\left((1,\ldots,n)\right)$ for some $\sigma \in S_n$, and so $c_{\lambda,A} = \sigma c_\lambda \sigma^{-1}$. Then

$$c^2_{\lambda,A}(e)c_{\lambda,A} = c^2_{\lambda,A} = \sigma c^2_\lambda \sigma^{-1} = c^2_\lambda(e)\sigma c_\lambda \sigma^{-1} = c^2_\lambda(e)c_{\lambda,A}.$$

But $c_{\lambda,A} \neq 0$ since $c_{\lambda,A}(e) = 1$. (Recall that $pq = e$ if and only if $p = q = e$ in $c_{\lambda,A} = \sum_{\substack{p\in P_{\lambda,A}\\ q\in Q_{\lambda,A}}}(\operatorname{sgn} q)pq$.) Hence $c^2_{\lambda,A}(e) = c^2_\lambda(e)$. □

The elements $c_{\lambda,A}$ are known as Young symmetrizers. Although neither Hermitian nor idempotent in general, they are used to construct such elements, which will in turn be used to describe the irreducible representations of S_n and their characters. The Young symmetrizers are also used to produce minimal left ideals in $\mathbb{C}S_n$.

Proposition 1. *The ideal $\mathcal{I}_{\lambda,A} := (\mathbb{C}S_n)\, c_{\lambda,A}$ is a minimal left ideal in $\mathbb{C}S_n$.*

Proof. Since $c_{\lambda,A} \neq 0$, $\mathcal{I}_{\lambda,A}$ is obviously a nonzero left ideal. We note that $c_{\lambda,A}\mathcal{I}_{\lambda,A} \subset \mathbb{C}c_{\lambda,A}$, since $c_{\lambda,A}xc_{\lambda,A} = mc_{\lambda,A}$ for some $m \in \mathbb{C}$ by Corollary 1. Let $\mathcal{J}$ be a left ideal contained in $\mathcal{I}_{\lambda,A}$. Then $c_{\lambda,A}\mathcal{J} \subset \mathbb{C}c_{\lambda,A}$, and both $c_{\lambda,A}\mathcal{J}$ and $\mathbb{C}c_{\lambda,A}$ are vector subspaces of $\mathbb{C}S_n$. However, $\mathbb{C}c_{\lambda,A}$ is one dimensional, and so we have only two possibilities.

Case 1. $c_{\lambda,A}\mathcal{J} = \mathbb{C}c_{\lambda,A}$. Then

$$\mathcal{I}_{\lambda,A} = (\mathbb{C}S_n)\,\mathbb{C}c_{\lambda,A} = (\mathbb{C}S_n)\,c_{\lambda,A}\mathcal{J} \subset \mathcal{J}$$

since $\mathcal{J}$ is a left ideal. Therefore $\mathcal{I}_{\lambda,A} = \mathcal{J}$.

Case 2. $c_{\lambda,A}\mathcal{J} = \{0\}$. Then

$$\mathcal{J}^2 \subset \mathcal{I}_{\lambda,A}\mathcal{J} = (\mathbb{C}S_n)\,c_{\lambda,A}\mathcal{J} = \{0\}.$$

But this implies $\mathcal{J} = \{0\}$, since $\mathbb{C}S_n$ is a symmetric algebra (possesses a nondegenerate involution). □

Theorem 1. *$\mathcal{I}_{\lambda,A}$ is invariant under the left regular representation π_{L} of S_n, and the restriction $\pi_{\lambda,A}$ of π_{L} to $\mathcal{I}_{\lambda,A}$ is irreducible.*

Proof. Immediate from Proposition 1 and earlier remarks about group algebras $\mathbb{C}G$. □

Theorem 2. *For $\lambda \neq \mu$ the representations $\pi_{\lambda,A}$ and $\pi_{\mu,B}$ are inequivalent.*

Proof. $a_{\lambda,A}\mathcal{I}_{\mu,B} = a_{\lambda,A}\left(\mathbb{C}S_n\right) a_{\mu,B} b_{\mu,B} \subset a_{\lambda,A}\left(\mathbb{C}S_n\right) b_{\mu,B} = \{0\}$, by Corollary 1. On the other hand,

$$a_{\lambda,A} c_{\lambda,A} = a_{\lambda,A}^2 b_{\lambda,A} = \lambda! a_{\lambda,A} b_{\lambda,A} = \lambda! c_{\lambda,A} \neq 0.$$

Suppose $\pi_{\lambda,A}$ and $\pi_{\mu,B}$ are equivalent as group representations. Then their corresponding algebra representations are also equivalent, so we can find an invertible linear map $T : \mathcal{I}_{\lambda,A} \twoheadrightarrow \mathcal{I}_{\mu,B}$ such that for all $x \in \mathbb{C}S_n$, $T \circ \hat{\pi}_{\lambda,A}(x) = \hat{\pi}_{\mu,B}(x) \circ T$. Hence we have

$$\begin{aligned}
T\left(a_{\lambda,A} c_{\lambda,A}\right) &= T \circ \hat{\pi}_{\lambda,A}\left(a_{\lambda,A}\right)\left(c_{\lambda,A}\right) \\
&= \hat{\pi}_{\mu,B}\left(a_{\lambda,A}\right) \circ T\left(c_{\lambda,A}\right) \\
&= a_{\lambda,A} T\left(c_{\lambda,A}\right) = 0 && \text{since } T(x) \in \mathcal{I}_{\mu,B} \text{ and } a_{\lambda,A}\mathcal{I}_{\mu,B} = \{0\} \\
\Rightarrow a_{\lambda,A} c_{\lambda,A} &= 0 && \text{since } T \text{ is invertible}
\end{aligned}$$

which is a contradiction. Therefore $\pi_{\lambda,A}$ and $\pi_{\mu,B}$ are inequivalent. □

Theorem 3. *$\pi_{\lambda,A}$ and $\pi_{\lambda,B}$ are equivalent.*

Proof. For some $\sigma \in S_n$ $\sigma(A) = B$, and so $\sigma c_{\lambda,A} \sigma^{-1} = c_{\lambda,B}$. Define $T : \mathcal{I}_{\lambda,A} \to \mathcal{I}_{\lambda,B}$ to be the linear map $x \mapsto x\sigma^{-1}$. T is well-defined, since if $x \in \mathcal{I}_{\lambda,A}$ then $x = y c_{\lambda,A}$ for some $y \in \mathbb{C}S_n$, and so $x\sigma^{-1} = \left(y\sigma^{-1}\right) \sigma c_{\lambda,A} \sigma^{-1} = y\sigma^{-1} c_{\lambda,B} \in \mathcal{I}_{\lambda,B}$. T is onto since $y c_{\lambda,B} = y\sigma c_{\lambda,A}\sigma^{-1}$ and $y\sigma c_{\lambda,A} \in \mathcal{I}_{\lambda,A}$, and T is obviously one-to-one. Finally we check the intertwining property: for $\tau \in S_n$ and $x \in \mathcal{I}_{\lambda,A}$,

$$\begin{aligned}
T \circ \pi_{\lambda,A}(\tau)(x) = T(\tau x) = \tau x \sigma^{-1} &= \pi_{\lambda,B}(\tau)\left(x\sigma^{-1}\right) \\
&= \pi_{\lambda,B}(\tau) \circ T(x).
\end{aligned}$$

□

It follows that the set $\{(\pi_\lambda, \mathcal{I}_\lambda) \,|\, \lambda \text{ is a partition of } n\}$ is a complete set of inequivalent irreducible representations for S_n. We may produce the ideals $\mathcal{I}_\lambda$ according to the following process.

(1) Associate to each conjugacy class in S_n the corresponding partition λ of n, and the canonical Young tableau Σ_λ.
(2) For each λ, calculate the subgroups P_λ, Q_λ and the Young symmetrizer c_λ for Σ_λ.
(3) $\{\sigma c_\lambda | \sigma \in S_n\}$ is a generating set for $\mathcal{I}_\lambda$ as a vector space, hence contains a basis $\beta_\lambda = \{\sigma_1 c_\lambda, \ldots, \sigma_{k_\lambda} c_\lambda\}$.
(4) The matrix of $\pi_\lambda(\sigma)$ in basis β_λ is then given by the equations defining the matrix coefficients $a_{ij}(\sigma)$:

$$\pi_\lambda(\sigma)\left(\sigma_j c_\lambda\right) = \sigma\sigma_j c_\lambda = \sum_{i=1}^{k_\lambda} a_{ij}(\sigma)\sigma_i c_\lambda,$$

so that

$$\left[\pi_\lambda(\sigma)\right]_{\beta_\lambda} = \left(a_{ij}(\sigma)\right)_{1\leq i,j\leq k_\lambda}.$$

(5) We may also orthonormalize basis β_λ to β_λ' with respect to the inner product in $\mathbb{C}S_n$. Then $\left[\pi_\lambda(\sigma)\right]_{\beta_\lambda'}$ will be a unitary matrix since the left regular representation of S_n is unitary with respect to this inner product.

Characters of irreducible representations of S_n. Since equivalent representations have the same character, we denote by χ_λ the character of the irreducible representation $(\pi_{\lambda,A}, \mathcal{I}_{\lambda,A})$ of S_n. We have seen in Lemma 2 that in $\mathbb{C}G$ in general the character of a subrepresentation (π, W) of the left regular representation of G is determined by the idempotent ϵ, where $W = (\mathbb{C}G)\epsilon$. Although $c_{\lambda,A}$ is not idempotent, the relation $c_{\lambda,A}^2 = c_\lambda^2(e) c_{\lambda,A}$ will allow easy definition of an idempotent generator of $\mathcal{I}_{\lambda,A}$ provided that $c_\lambda^2(e) \neq 0$.

Lemma 4. *Define $m_\lambda = c_\lambda^2(e)$. Then $m_\lambda = n!/k_\lambda$, where k_λ is the vector space dimension of $\mathcal{I}_\lambda$.*

Proof. Define the linear operator T_λ on $\mathbb{C}S_n$ by $T_\lambda(x) = xc_\lambda$. Then $\mathcal{I}_\lambda$ is T_λ-invariant, and $T_\lambda(xc_\lambda) = xc_\lambda^2 = m_\lambda x c_\lambda$, so that the restriction of T_λ to $\mathcal{I}_\lambda$ is the operator $m_\lambda \mathbf{1}_{\mathcal{I}_\lambda}$. Choose any basis of $\mathcal{I}_\lambda$ and extend it to a basis of β of $\mathbb{C}S_n$. Then

$$[T_\lambda]_\beta = \left[\begin{array}{c|c} m_\lambda \mathbf{I}_{k_\lambda} & * \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right]$$

where $\mathbf{0}$ represents a zero block of appropriate dimensions, $*$ is an arbitrary block, and $\mathbf{I}_{k_\lambda}$ is the $k_\lambda \times k_\lambda$ identity matrix. Hence $\operatorname{tr} T_\lambda = m_\lambda k_\lambda$.

On the other hand in the basis $\alpha = \{\sigma_1, \ldots, \sigma_{n!}\}$ of $\mathbb{C}S_n$ formed by ordering the elements of S_n in some fashion, we have

$$T_\lambda(\sigma_j) = \sigma_j c_\lambda = \sum_{i=1}^{n!} c_\lambda\left(\sigma_j^{-1}\sigma_i\right)\sigma_i$$

and so

$$[T_\lambda]_\alpha = \left(c_\lambda\left(\sigma_j^{-1}\sigma_i\right)\right)_{1 \le i,j \le n!}.$$

Hence

$$\operatorname{tr} T_\lambda = \sum_{i=1}^{n!} c_\lambda\left(\sigma_i^{-1}\sigma_i\right) = c_\lambda(e) n! = n!$$

since $c_\lambda(e) = 1$, by straightforward calculation. □

In particular $m_\lambda \neq 0$, and so we set $\epsilon_{\lambda,A} := m_\lambda^{-1} c_{\lambda,A}$. We find

$$\epsilon_{\lambda,A}^2 = m_\lambda^{-2} c_{\lambda,A}^2 = m_\lambda^{-2} m_\lambda c_{\lambda,A} = \epsilon_{\lambda,A}$$

and obviously $\mathcal{I}_{\lambda,A} = (\mathbb{C}S_n)\,\epsilon_{\lambda,A}$. Then the following theorem is an immediate application of the formula for a character in terms of an idempotent generator of its representation space, when this is a subspace of $\mathbb{C}G$, described in Lemma 2.

Theorem 4. *The character χ_λ of the irreducible representation $(\pi_{\lambda,A}, \mathcal{I}_{\lambda,A})$ of S_n is given by*

$$(\dagger) \qquad \chi_\lambda(\sigma) = \sum_{\tau \in S_n} \epsilon_{\lambda,A}\left(\tau\sigma^{-1}\tau^{-1}\right) = \frac{k_\lambda}{n!} \sum_{\tau \in S_n} c_{\lambda,A}\left(\tau\sigma^{-1}\tau^{-1}\right)$$

where $k_\lambda = \dim_{\mathbb{C}} \mathcal{I}_{\lambda,A}$. (The sums are independent of the choice of A.)

From Theorem 4 is derived the Frobenius formula for calculating χ_λ and a related result for finding k_λ. The Murnaghan-Nakayama rule gives an inductive procedure for calculating characters in terms of Young diagrams. However, these results will not be considered here. See for example [Fulton] for details.

Decomposition of representations of S_n into direct sums of irreducibles. For each partition λ of n we define the element $\omega_\lambda := m_\lambda^{-1} \sum_{\sigma \in S_n} \sigma \epsilon_\lambda \sigma^{-1}$. ω_λ is independent of the choice of Young tableau $\Sigma_{\lambda,A}$ for λ, since as σ ranges through S_n, $\sigma \epsilon_\lambda \sigma^{-1}$ ranges through $\epsilon_{\lambda,A}$ for all $n!$ possible Young tableaux $\Sigma_{\lambda,A}$ corresponding to the diagram Σ_λ.

We find

$$\text{(‡)} \qquad \begin{aligned} \omega_\lambda(\tau) &= \frac{1}{m_\lambda} \sum_{\sigma \in S_n} \epsilon_\lambda \left(\sigma^{-1} \tau \sigma\right) = \frac{1}{m_\lambda} \sum_{\sigma \in S_n} \epsilon_\lambda \left(\sigma \tau \sigma^{-1}\right) \\ &= \frac{1}{m_\lambda} \chi_\lambda \left(\tau^{-1}\right) \qquad \text{by (†)}. \end{aligned}$$

Proposition 2. *ω_λ has the following properties.*

1. $\omega_\lambda \neq 0$.
2. ω_λ *is Hermitian and idempotent.*
3. $\omega_\lambda \in \mathrm{Z}\left(\mathbb{C}S_n\right)$.
4. $\omega_\lambda \omega_\mu = 0$ *for* $\lambda \neq \mu$ *and* $\langle \omega_\lambda, \omega_\mu \rangle = \delta_{\lambda,\mu}/m_\lambda^2$.
5. *The set* $\{\omega_\lambda | \lambda$ *is a partition of* $n\}$ *is a basis of* $\mathrm{Z}\left(\mathbb{C}S_n\right)$.

Proof.

(1)

$$\text{(§)} \qquad \omega_\lambda(e) = \frac{1}{m_\lambda} \chi_\lambda(e) = \frac{k_\lambda}{m_\lambda} = \frac{k_\lambda^2}{n!} \neq 0 \qquad \text{by (‡) and Lemma 4}$$

so $\omega_\lambda \neq 0$.

(2) Since the left regular representation π_{L} is unitary with respect to the inner product we have in $\mathbb{C}S_n$, it follows that π_λ is unitary in $\mathcal{I}_\lambda$ with the same inner product. Hence $\chi_\lambda\left(\sigma^{-1}\right) = \overline{\chi_\lambda(\sigma)}$. Then

$$\begin{aligned} \omega_\lambda^* &= \sum_{\sigma \in S_n} \overline{\omega_\lambda(\sigma)} \sigma^{-1} = \frac{1}{m_\lambda} \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \sigma^{-1} \\ &= \frac{1}{m_\lambda} \sum_{\sigma \in S_n} \chi_\lambda\left(\sigma^{-1}\right) \sigma = \sum_{\sigma \in S_n} \omega_\lambda(\sigma) \sigma \\ &= \omega_\lambda \end{aligned}$$

using equation (‡), so ω_λ is Hermitian.

To show that ω_λ is idempotent, we compute using the matrix coefficients $a_{i,j}$ of π_λ with respect to a fixed basis of $\mathcal{I}_\lambda$. We show idempotence by comparing coefficients in ω_λ^2, ω_λ:

$$\begin{aligned} \omega_\lambda^2(\sigma) &= \sum_{\tau \in S_n} \omega_\lambda(\tau) \omega_\lambda\left(\tau^{-1} \sigma\right) \\ &= \frac{1}{m_\lambda^2} \sum_{\tau \in S_n} \chi_\lambda(\tau^{-1}) \chi_\lambda\left(\sigma^{-1} \tau\right) \quad \text{by (‡)}. \end{aligned}$$

But

$$\begin{aligned}\chi_\lambda(\tau^{-1})\chi_\lambda\left(\sigma^{-1}\tau\right) &= \overline{\operatorname{tr}\pi_\lambda(\tau)}\operatorname{tr}\left(\pi_\lambda\left(\sigma^{-1}\right)\pi_\lambda(\tau)\right)\\ &= \sum_{i=1}^{k_\lambda}\overline{a_{i,i}(\tau)}\sum_{j=1}^{k_\lambda}\sum_{k=1}^{k_\lambda}a_{j,k}\left(\sigma^{-1}\right)a_{k,j}(\tau)\\ &= \sum_{j,k=1}^{k_\lambda}a_{j,k}\left(\sigma^{-1}\right)\sum_{i=1}^{k_\lambda}a_{k,j}(\tau)\overline{a_{i,i}(\tau)}.\end{aligned}$$

Hence

$$\begin{aligned}\omega_\lambda^2(\sigma) &= \frac{1}{m_\lambda^2}\sum_{j,k=1}^{k_\lambda}a_{j,k}\left(\sigma^{-1}\right)\\ &\quad\cdot\sum_{i=1}^{k_\lambda}\sum_{\tau\in S_n}a_{k,j}(\tau)\overline{a_{i,i}(\tau)}\\ &= \frac{n!}{m_\lambda^2}\sum_{j,k=1}^{k_\lambda}a_{j,k}\left(\sigma^{-1}\right)\sum_{i=1}^{k_\lambda}\left(a_{k,j},a_{i,i}\right) && \text{(the inner product of } \mathcal{A}\left(S_n\right))\\ &= \frac{1}{m_\lambda}\sum_{j,k=1}^{k_\lambda}a_{j,k}\left(\sigma^{-1}\right)\sum_{i=1}^{k_\lambda}\delta_{k,i}\delta_{j,i} && \text{orthogonality relations, and } k_\lambda = n!/m_\lambda \text{ by Lemma 4}\\ &= \frac{1}{m_\lambda}\sum_{i=1}^{k_\lambda}a_{i,i}\left(\sigma^{-1}\right)\\ &= \frac{1}{m_\lambda}\chi_\lambda\left(\sigma^{-1}\right) = \omega_\lambda\left(\sigma\right) && \text{by } (\ddagger).\end{aligned}$$

Therefore ω_λ is idempotent.

(3) It is clear that ω_λ is a class function, and so $\omega_\lambda \in \mathrm{Z}\left(\mathbb{C}S_n\right)$.

(4)

$$\begin{aligned}\omega_\lambda\omega_\mu &= \frac{1}{m_\lambda m_\mu}\sum_{\sigma,\tau\in S_n}\sigma\omega_\lambda\sigma^{-1}\tau\omega_\mu\tau^{-1}\\ &= \sum_{\sigma,\tau\in S_n}c_{\lambda,\sigma((1,\dots,n))}c_{\mu,\tau((1,\dots,n))}\\ &= 0 && \text{if } \lambda\neq\mu, \text{ by Corollary 1.}\end{aligned}$$

Then

$$\begin{aligned}\langle\omega_\lambda,\omega_\mu\rangle &= \frac{1}{n!}\omega_\mu^*\omega_\lambda(e) = \frac{1}{n!}\omega_\mu\omega_\lambda(e) && \text{by (2)}\\ &= \frac{1}{n!}\delta_{\lambda,\mu}\omega_\lambda^2(e) = \frac{1}{n!}\delta_{\lambda,\mu}\omega_\lambda(e) && \text{by (2)}\\ &= \delta_{\lambda,\mu}\frac{k_\lambda^2}{(n!)^2} = \frac{\delta_{\lambda,\mu}}{m_\lambda^2} && \text{by (§) and Lemma 4.}\end{aligned}$$

(5) Since $\{\omega_\lambda | \lambda \text{ is a partition of } n\}$ is an orthogonal set of nonzero elements in $\mathrm{Z}\left(\mathbb{C}S_n\right)$ by (1), (3) and (4), with number of elements equal to the number of conjugacy classes in S_n, it forms a basis of $\mathrm{Z}\left(\mathbb{C}S_n\right)$. □

Now $\{(\pi_\lambda, \mathcal{I}_\lambda) \,|\, \lambda \text{ is a partition of } n\}$ is a complete set of irreducible representations for S_n. Let (π, V) be any finite dimensional representation of S_n. Then we have the projections

$$\begin{aligned} P_\lambda &= (\dim_{\mathbb{C}} \mathcal{I}_\lambda)\, |S_n|^{-1} \sum_{\sigma \in S_n} \overline{\chi_\lambda(\sigma)} \pi(\sigma) \\ &= \frac{1}{m_\lambda} \sum_{\sigma \in S_n} \chi_\lambda\left(\sigma^{-1}\right) \pi(\sigma) \\ &= \hat{\pi}\left(\omega_\lambda\right). \end{aligned}$$

It is then evident, from the general results for $\mathbb{C}G$, that

$$\hat{\pi}\left(\omega_\lambda\right) V \simeq \mathcal{I}_\lambda^{\oplus r_\lambda}$$

and

$$\pi \left|_{\hat{\pi}(\omega_\lambda) V} \right. \simeq \pi_\lambda^{\oplus r_\lambda}$$

for some nonnegative integer r_λ. Therefore

$$V = \bigoplus_\lambda \hat{\pi}\left(\omega_\lambda\right) V \simeq \bigoplus_\lambda \mathcal{I}_\lambda^{\oplus r_\lambda}$$

and

$$\pi \simeq \bigoplus_\lambda \pi_\lambda^{\oplus r_\lambda}.$$

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