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Linear Algebraic Groups

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ABSTRACT. We give a summary, without proofs, of basic properties of linear algebraic groups, with particular emphasis on reductive algebraic groups.

1. Algebraic groups

Let K be an algebraically closed field. An *algebraic K-group* $\mathbf{G}$ is an algebraic variety over K, and a group, such that the maps $\mu : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$, $\mu(x,y) = xy$, and $\iota : \mathbf{G} \to \mathbf{G}$, $\iota(x) = x^{-1}$, are morphisms of algebraic varieties. For convenience, in these notes, we will fix K and refer to an algebraic K-group as an algebraic group. If the variety $\mathbf{G}$ is affine, that is, $\mathbf{G}$ is an algebraic set (a Zariski-closed set) in K^n for some natural number n, we say that $\mathbf{G}$ is a *linear algebraic group.* If $\mathbf{G}$ and $\mathbf{G}'$ are algebraic groups, a map $\varphi : \mathbf{G} \to \mathbf{G}'$ is a *homomorphism of algebraic groups* if φ is a morphism of varieties and a group homomorphism. Similarly, φ is an *isomorphism of algebraic groups* if φ is an isomorphism of varieties and a group isomorphism.

A closed subgroup of an algebraic group is an algebraic group. If $\mathbf{H}$ is a closed subgroup of a linear algebraic group $\mathbf{G}$, then $\mathbf{G}/\mathbf{H}$ can be made into a quasi-projective variety (a variety which is a locally closed subset of some projective space). If $\mathbf{H}$ is normal in $\mathbf{G}$, then $\mathbf{G}/\mathbf{H}$ (with the usual group structure) is a linear algebraic group.

Let $\varphi : \mathbf{G} \to \mathbf{G}'$ be a homomorphism of algebraic groups. Then the kernel of φ is a closed subgroup of $\mathbf{G}$ and the image of φ is a closed subgroup of $\mathbf{G}$.

Let X be an affine algebraic variety over K, with affine algebra (coordinate ring) $K[X] = K[x_1, \ldots, x_n]/I$. If k is a subfield of K, we say that X is *defined over k* if the ideal I is generated by polynomials in $k[x_1, \ldots, x_n]$, that is, I is generated by $I_k := I \cap k[x_1, \ldots, x_n]$. In this case, the k-subalgebra $k[X] := k[x_1, \ldots, x_n]/I_k$ of $K[X]$ is called a *k-structure* on X, and $K[X] = k[X] \otimes_k K$. If X and X' are algebraic varieties defined over k, a morphism $\varphi : X \to X'$ is *defined over k* (or is a *k-morphism*) if there is a homomorphism $\varphi_k^* : k[X'] \to k[X]$ such that the algebra homomorphism $\varphi^* : K[X'] \to K[X]$ defining φ is $\varphi_k^* \times id$. Equivalently, the coordinate functions of φ all have coefficients in k. The set $X(k) := X \cap k^n$ is called the *K-rational* points of X.

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If k is a subfield of K, we say that a linear algebraic group $\mathbf{G}$ is *defined over* k (or is a k-*group*) if the variety $\mathbf{G}$ is defined over k and the homomorphisms μ and ι are defined over k. Let $\varphi : \mathbf{G} \to \mathbf{G}'$ be a k-homomorphism of k-groups. Then the image of φ is defined over k but the kernel of φ might not be defined over k.

An algebraic variety X over K is *irreducible* if it cannot be expressed as the union of two proper closed subsets. Any algebraic variety X over K can be expressed as the union of finitely many irreducible closed subsets:

$$X = X_1 \cup X_2 \cup \cdots \cup X_r,$$

where $X_i \not\subset X_j$ if $j \neq i$. This decomposition is unique and the X_i are the maximal irreducible subsets of X (relative to inclusion). The X_i are called the *irreducible components* of X.

Let $\mathbf{G}$ be an algebraic group. Then $\mathbf{G}$ has a unique irreducible component $\mathbf{G}^0$ containing the identity element. The irreducible component $\mathbf{G}^0$ is a closed normal subgroup of $\mathbf{G}$. The cosets of $\mathbf{G}^0$ in $\mathbf{G}$ are the irreducible components of $\mathbf{G}$, and $\mathbf{G}^0$ is the connected component of the identity in $\mathbf{G}$. Also, if $\mathbf{H}$ is a closed subgroup of $\mathbf{G}$ of finite index in $\mathbf{G}$, then $\mathbf{H} \supset \mathbf{G}^0$. For a linear algebraic group, connectedness is equivalent to irreducibility. It is usual to refer to an irreducible algebraic group as a connected algebraic group.

If $\varphi : \mathbf{G} \to \mathbf{G}'$ is a homomorphism of algebraic groups, then $\varphi(\mathbf{G}^0) = \varphi(\mathbf{G})^0$. If k is a subfield of K and $\mathbf{G}$ is defined over k, then $\mathbf{G}^0$ is defined over k.

The *dimension* of $\mathbf{G}$ is the dimension of the variety $\mathbf{G}^0$. That is, the dimension of $\mathbf{G}$ is the transcendence degree of the field $K(\mathbf{G}^0)$ over K.

If $\mathbf{G}$ is a linear algebraic group, then $\mathbf{G}$ is isomorphic, as an algebraic group, to a closed subgroup of $\mathbf{GL}_n(K)$ for some natural number n.

EXAMPLE 1.1. $\mathbf{G} = K$, with $\mu(x, y) = x+y$ and $\iota(x) = -x$. The usual notation for this group is $\mathbf{G}_a$. It is connected and has dimension 1.

EXAMPLE 1.2. Let n be a positive integer and let $M_n(K)$ be the set of $n \times n$ matrices with entries in K. The general linear group $\mathbf{G} = \mathbf{GL}_n(K)$ is the group of matrices in $M_n(K)$ that have nonzero determinant. Note that $\mathbf{G}$ can be identified with the closed subset $\{(g, x) \mid g \in M_n(K),\ x \in K,\ (\det g)x = 1\}$ of $K^{n^2} \times K = K^{n^2+1}$. Then $K[\mathbf{G}] = K[\,x_{ij},\, 1 \leq i, j \leq n,\, \det(x_{ij})^{-1}]$. The dimension of $\mathbf{GL}_n(K)$ is n^2, and it is connected. In the case $n = 1$, the usual notation for $\mathbf{GL}_1(K)$ is $\mathbf{G}_m$. The only connected algebraic groups of dimension 1 are $\mathbf{G}_a$ and $\mathbf{G}_m$.

EXAMPLE 1.3. Let n be a positive integer and let I_n be the $n \times n$ identity matrix. The $2n \times 2n$ matrix $J = \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right]$ is invertible and satisfies ${}^tJ = -J$, where ${}^tJ$ denotes the transpose of J. The $2n \times 2n$ symplectic group $\mathbf{G} = \mathbf{Sp}_{2n}(K)$ is defined by $\{\, g \in M_{2n}(K) \mid {}^tgJg = J \,\}$.

2. Jordan decomposition in linear algebraic groups

Recall that a matrix $x \in M_n(K)$ is *semisimple* if x is diagonalizable: there is a $g \in \mathbf{GL}_n(K)$ such that gxg^{-1} is a diagonal matrix. Also, x is unipotent if $x - I_n$ is nilpotent: $(x - I_n)^k = 0$ for some natural number k. Given $x \in \mathbf{GL}_n(K)$, there exist elements x_s and x_u in $\mathbf{GL}_n(K)$ such that x_s is semisimple, x_u is unipotent, and $x = x_s x_u = x_u x_s$. Furthermore, x_s and x_u are uniquely determined.

Now suppose that $\mathbf{G}$ is a linear algebraic group. Choose n and an injective homomorphism $\varphi : \mathbf{G} \to \mathbf{GL}_n(K)$ of algebraic groups. If $g \in \mathbf{G}$, the semisimple and unipotent parts $\varphi(g)_s$ and $\varphi(g)_u$ of $\varphi(g)$ lie in $\varphi(\mathbf{G})$, and the elements g_s and g_u such that $\varphi(g_s) = \varphi(g)_s$ and $\varphi(g_u) = \varphi(g)_u$ depend only on g and not on the choice of φ (or n). The elements g_s and g_u are called the semisimple and unipotent part of g, respectively. An element $g \in \mathbf{G}$ is *semisimple* if $g = g_s$ (and $g_u = 1$), and *unipotent* if $g = g_u$ (and $g_s = 1$).

Jordan decomposition.

(1) If $g \in \mathbf{G}$, there exist elements g_s and g_u in $\mathbf{G}$ such that $g = g_s g_u = g_u g_s$, g_s is semisimple, and g_u is unipotent. Furthermore, g_s and g_u are uniquely determined by the above conditions.

(2) If k is a perfect subfield of K and $\mathbf{G}$ is a k-group, then $g \in \mathbf{G}(k)$ implies $g_s,\ g_u \in \mathbf{G}(k)$.

Jordan decompositions are preserved by homomorphisms of algebraic groups. Suppose that $\mathbf{G}$ and $\mathbf{G}'$ are linear algebraic groups and $\varphi : \mathbf{G} \to \mathbf{G}'$ is a homomorphism of linear algebraic groups. Let $g \in \mathbf{G}$. Then $\varphi(g)_s = \varphi(g_s)$ and $\varphi(g)_u = \varphi(g_u)$.

3. Lie algebras

Let $\mathbf{G}$ be a linear algebraic group. The tangent bundle $T(\mathbf{G})$ of $\mathbf{G}$ is the set $\mathrm{Hom}_{K-alg}(K[\mathbf{G}], K[t]/(t^2))$ of K-algebra homomorphisms from the affine algebra $K[\mathbf{G}]$ of $\mathbf{G}$ to the algebra $K[t]/(t^2)$. If $g \in \mathbf{G}$, the evaluation map $f \mapsto f(g)$ from $K[\mathbf{G}]$ to K is a K-algebra isomorphism. This results in a bijection betweeen $\mathbf{G}$ and $\mathrm{Hom}_{K-alg}(K[\mathbf{G}], K)$. Composing elements of $T(\mathbf{G})$ with the map $a + bt + (t^2) \mapsto a$ from $K[t]/(t^2)$ to K results in a map from $T(\mathbf{G})$ to $\mathbf{G} = \mathrm{Hom}_{K-alg}(K[\mathbf{G}], K)$. The tangent space $T_1(\mathbf{G})$ of $\mathbf{G}$ at the identity element 1 of $\mathbf{G}$ is the fibre of $T(\mathbf{G})$ over 1. If $X \in T_1(\mathbf{G})$ and $f \in K[\mathbf{G}]$, then $X(f) = f(1) + t\, d_X(f) + (t^2)$ for some $d_X(f) \in K$. This defines a map $d_X : K[\mathbf{G}] \to K$ which satisfies:

$$d_X(f_1 f_2) = d_X(f_1) f_2(1) + f_1(1) d_X(f_2), \qquad f_1,\ f_2 \in K[\mathbf{G}].$$

Let $\mu^* : K[\mathbf{G}] \to K[\mathbf{G}] \otimes_K K[\mathbf{G}]$ be the K-algebra homomorphism which corresponds to the multiplication map $\mu : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$. Set $\delta_X = (1 \otimes d_X) \circ \mu^*$. The map $\delta_X : K[\mathbf{G}] \to K[\mathbf{G}]$ is a K-linear map and a *derivation*:

$$\delta_X(f_1 f_2) = \delta_X(f_1) f_2 + f_1 \delta_X(f_2), \qquad f_1,\ f_2 \in K[\mathbf{G}].$$

Furthermore, δ_X is left-invariant: $\ell_g \delta_X = \delta_X \ell_g$ for all $g \in \mathbf{G}$, where $(\ell_g f)(g') = f(g^{-1} g')$, $f \in K[\mathbf{G}]$. The map $X \mapsto \delta_X$ is a K-linear isomorphism of $T_1(\mathbf{G})$ onto the vector space of K-linear maps from $K[\mathbf{G}]$ to $K[\mathbf{G}]$ which are left-invariant derivations.

Let $\mathfrak{g} = T_1(\mathbf{G})$. Define $[X, Y] \in \mathfrak{g}$ by $\delta_{[X,Y]} = \delta_X \circ \delta_Y - \delta_Y \circ \delta_X$. Then $\mathfrak{g}$ is a vector space over K and the map $[\cdot, \cdot]$ satisfies:

(1) $[\cdot, \cdot]$ is linear in both variables
(2) $[X, X] = 0$ for all $X \in \mathfrak{g}$
(3) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ for all $X,\ Y,\ X \in \mathfrak{g}$. (*Jacobi identity*)

Therefore $\mathfrak{g}$ is a Lie algebra over K. We call it the Lie algebra of $\mathbf{G}$.

Example 3.1. If $\mathbf{G} = \mathbf{GL}_n(K)$, then $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{gl}_n(K)$ which is $M_n(K)$ equipped with the Lie bracket $[X, Y] = XY - YX$, $X,\ Y \in M_n(K)$.

EXAMPLE 3.2. If $\mathbf{G} = \mathbf{Sp}_{2n}(K)$, then $\mathfrak{g}$ is isomorphic to the Lie algebra $\{ X \in M_{2n}(K) \mid {}^tXJ + JX = 0 \}$, with bracket $[X,Y] = XY - YX$.

Let $\varphi : \mathbf{G} \to \mathbf{G}'$ be a homomorphism of linear algebraic groups. Composition with the algebra homomorphism $\varphi^* : K[\mathbf{G}'] \to K[\mathbf{G}]$ results in a map $T(\varphi) : T(\mathbf{G}) \to T(\mathbf{G}')$. The *differential* $d\varphi$ of φ is the restriction $d\varphi = T(\varphi)\,|_{\mathfrak{g}}$ of $T(\varphi)$ to $\mathfrak{g}$. It is a K-linear map from $\mathfrak{g}$ to $\mathfrak{g}'$, and satisfies

$$d\varphi([X,Y]) = [d\varphi(X), d\varphi(Y)], \qquad X,\, Y \in \mathfrak{g}.$$

That is, $d\varphi$ is a *homomorphism of Lie algebras.* If φ is bijective, then φ is an isomorphism if and only if $d\varphi$ is an isomorphism of Lie algebras. If K has characteristic zero, any bijective homomorphism of linear algebraic groups is an isomorphism.

If $\mathbf{H}$ is a closed subgroup of a linear algebraic group $\mathbf{G}$, then (via the differential of inclusion) the Lie algebra $\mathfrak{h}$ of $\mathbf{H}$ is isomorphic to a Lie subalgebra of $\mathfrak{g}$. And $\mathbf{H}$ is a normal subgroup of $\mathbf{G}$ if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ ($[X,Y] \in \mathfrak{h}$ whenever $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$).

If $g \in \mathbf{G}$, then $\mathrm{Int}_g : \mathbf{G} \to \mathbf{G}$, $\mathrm{Int}_g = g g_0 g^{-1}$, $g_0 \in \mathbf{G}$, is an isomorphism of algebraic groups, so $\mathrm{Ad}\, g := d(\mathrm{Int}_g) : \mathfrak{g} \to \mathfrak{g}$ is an isomorphism of Lie algebras. Note that $(\mathrm{Ad}\, g)^{-1} = \mathrm{Ad}\, g^{-1}$, $g \in \mathbf{G}$, and $\mathrm{Ad}\,(g_1 g_2) = \mathrm{Ad}\, g_1 \circ \mathrm{Ad}\, g_2$, $g_1,\, g_2 \in \mathbf{G}$. The map $\mathrm{Ad} : \mathbf{G} \to \mathbf{GL}(\mathfrak{g})$ is a homomorphism of algebraic groups, called the *adjoint representation* of $\mathbf{G}$.

If $\mathbf{G}$ is a k-group, then its Lie algebra $\mathfrak{g}$ has a natural k-structure $\mathfrak{g}(k)$, with $\mathfrak{g} \simeq K \otimes_k \mathfrak{g}(k)$. Also, Ad is defined over k.

Jordan decomposition in the Lie algebra. We can define semisimple and nilpotent elements in $\mathfrak{g}$ in manner analogous to definitions of semisimple and unipotent elements in $\mathbf{G}$ (as $\mathfrak{g}$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}_n(K)$ for some n). If $X \in \mathfrak{g}$, there exist unique elements X_s and $X_n \in \mathfrak{g}$ such that $X = X_s + X_n$, $[X_s, X_n] = 0$, X_s is semisimple, and X_n is nilpotent. If $\varphi : \mathbf{G} \to \mathbf{G}'$ is a homomorphism of algebraic groups, then $d\varphi(X)_s = d\varphi(X_s)$ and $d\varphi(X)_n = d\varphi(X_n)$ for all $X \in \mathfrak{g}$.

4. Tori

A *torus* is a linear algebraic group which is isomorphic to the direct product $\mathbf{G}_m^d = \mathbf{G}_m \times \cdots \times \mathbf{G}_m$ (d times), where d is a positive integer. A linear algebraic group $\mathbf{G}$ is a torus if and only if $\mathbf{G}$ is connected and abelian, and every element of $\mathbf{G}$ is semisimple.

A *character* of a torus $\mathbf{T}$ is a homomorphism of algebraic groups from $\mathbf{T}$ to $\mathbf{G}_m$. The product of two characters of $\mathbf{T}$ is a character of $\mathbf{T}$, the inverse of a character of $\mathbf{T}$ is a character of $\mathbf{T}$, and characters of $\mathbf{T}$ commute with each other, so the set $X(\mathbf{T})$ of characters of $\mathbf{T}$ is an abelian group. A *one-parameter* subgroup of $\mathbf{T}$ is a homomorphism of algebraic groups from $\mathbf{G}_m$ to $\mathbf{T}$. The set $Y(\mathbf{T})$ of one-parameter subgroups is an abelian group. If $\mathbf{T} \simeq \mathbf{G}_m$, then $X(\mathbf{T}) = Y(\mathbf{T})$ is just the set of maps $x \mapsto x^r$, as r varies over $\mathbb{Z}$. In general, $\mathbf{T} \simeq \mathbf{G}_m^d$ for some positive integer d, so $X(\mathbf{T}) \simeq X(\mathbf{G}_m)^d \simeq \mathbb{Z}^d \simeq Y(\mathbf{T})$. We have a pairing

$$\begin{aligned} &\langle \cdot, \cdot \rangle : X(\mathbf{T}) \times Y(\mathbf{T}) \to \mathbb{Z} \\ &\langle \chi, \eta \rangle \mapsto r \text{ where } \chi \circ \eta(x) = x^r,\ x \in \mathbf{G}_m. \end{aligned}$$

Let k be a subfield of K. A torus $\mathbf{T}$ is a *k-torus* if $\mathbf{T}$ is defined over k. Let $\mathbf{T}$ be a k-torus. Let $X(\mathbf{T})_k$ be the subgroup of $X(\mathbf{T})$ made up of those characters of $\mathbf{T}$ which are defined over k. We say that $\mathbf{T}$ is *k-split* (or *splits over k*) whenever $X(\mathbf{T})_k$ spans $k[\mathbf{T}]$, or, equivalently, whenever $\mathbf{T}$ is k-isomorphic to $\mathbf{G}_m \times \cdots \times \mathbf{G}_m$ (d times, $d = \dim \mathbf{T}$). In this case, $\mathbf{T}(k) \simeq k^\times \times \cdots \times k^\times$. If $X(\mathbf{T})_k = 0$, then we say that $\mathbf{T}$ is *k-anisotropic.* There exists a finite Galois extension of k over which $\mathbf{T}$ splits. There exist unique tori $\mathbf{T}_{spl}$ and $\mathbf{T}_{an}$ of $\mathbf{T}$, both defined over k, such that $\mathbf{T} = \mathbf{T}_{spl}\mathbf{T}_{an}$, $\mathbf{T}_{spl}$ is k-split and $\mathbf{T}_{an}$ is k-anisotropic. Also, $\mathbf{T}_{an}$ is the identity component of $\bigcap_{\chi\in X(\mathbf{T})_k} \ker \chi$.

Example 4.1. Let $\mathbf{T}$ be the subgroup of $\mathbf{GL}_n(K)$ consisting of diagonal matrices in $\mathbf{GL}_n(K)$. Then $\mathbf{T}$ is a k-split k-torus for any subfield k of K.

Example 4.2. Let $\mathbf{T}$ be the closed subgroup of $\mathbf{GL}_2(\mathbb{C})$ defined by

$$\mathbf{T} = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{C},\ a^2 + b^2 \neq 0 \right\}.$$

Then $\mathbf{T}$ is an $\mathbb{R}$-torus and is $\mathbb{R}$-anisotropic.

5. Reductive groups, root systems and root data–the absolute case

Let $\mathbf{G}$ be a linear algebraic group which contains at least one torus. Then the set of tori in $\mathbf{G}$ has maximal elements, relative to inclusion. Such maximal elements are called *maximal tori* of $\mathbf{G}$. All of the maximal tori in $\mathbf{G}$ are conjugate. The *rank* of $\mathbf{G}$ is defined to be the dimension of a maximal torus in $\mathbf{G}$.

Now suppose that $\mathbf{G}$ is a linear algebraic group and $\mathbf{T}$ is a torus in $\mathbf{G}$. Recall that the adjoint representation $\mathrm{Ad} : \mathbf{G} \to \mathbf{GL}(\mathfrak{g})$ is a homomorphism of algebraic groups. Therefore $\mathrm{Ad}\,(\mathbf{T})$ consists of commuting semisimple elements, and so is diagonalizable. Given $\alpha \in X(\mathbf{T})$, let $\mathfrak{g}_\alpha = \{\, X \in \mathfrak{g} \mid \mathrm{Ad}\,(t)X = \alpha(t)X,\ \forall\, t \in \mathbf{T}\,\}$. The nonzero $\alpha \in X(\mathbf{T})$ such that $\mathfrak{g}_\alpha \neq 0$ are the *roots* of $\mathbf{G}$ relative to $\mathbf{T}$. The set of roots of $\mathbf{G}$ relative to $\mathbf{T}$ will be denoted by $\Phi(\mathbf{G}, \mathbf{T})$.

The centralizer $Z_{\mathbf{G}}(\mathbf{T})$ of $\mathbf{T}$ in $\mathbf{G}$ is the identity component of the normalizer $N_{\mathbf{G}}(\mathbf{T})$ of $\mathbf{T}$ in $\mathbf{G}$. The *Weyl group* $W(\mathbf{G}, \mathbf{T})$ of $\mathbf{T}$ in $\mathbf{G}$ is the (finite) quotient $N_{\mathbf{G}}(\mathbf{T})/Z_{\mathbf{G}}(\mathbf{T})$. Because $W(\mathbf{G}, \mathbf{T})$ acts on $\mathbf{T}$, $W(\mathbf{G}, \mathbf{T})$ also acts on $X(\mathbf{T})$, and $W(\mathbf{G}, \mathbf{T})$ permutes the roots of $\mathbf{T}$ in $\mathbf{G}$. Since any two maximal tori in $\mathbf{G}$ are conjugate, their Weyl groups are isomorphic. The Weyl group of any maximal torus is referred to as the Weyl group of $\mathbf{G}$.

An algebraic group $\mathbf{G}$ contains a unique maximal normal solvable subgroup, and this subgroup is closed. Its identity component is called the *radical* of $\mathbf{G}$, written $R(\mathbf{G})$. The set $R_u(\mathbf{G})$ of unipotent elements in $R(\mathbf{G})$ is a normal closed subgroup of $\mathbf{G}$, and is called the *unipotent radical* of $\mathbf{G}$. If $\mathbf{G}$ is a linear algebraic group such that the radical $R(\mathbf{G}^0)$ of $\mathbf{G}^0$ is trivial, then $\mathbf{G}$ is *semisimple.* In fact, $\mathbf{G}$ is semisimple if and only if $\mathbf{G}$ has no nontrivial connected abelian normal subgroups. If $R_u(\mathbf{G}^0)$ is trivial, then $\mathbf{G}$ is *reductive.* The *semisimple rank* of $\mathbf{G}$ is defined to be the rank of $\mathbf{G}/R(\mathbf{G})$, and the *reductive rank* of $\mathbf{G}$ is the rank of $\mathbf{G}/R_u(\mathbf{G})$.

The derived group $\mathbf{G}_{der}$ of $\mathbf{G}$ is a closed subgroup of $\mathbf{G}$, and is connected when $\mathbf{G}$ is connected. Suppose that $\mathbf{G}$ is connected and reductive. Then

(1) $\mathbf{G}_{der}$ is semisimple.
(2) $R(\mathbf{G}) = Z(\mathbf{G})^0$, where $Z(\mathbf{G})$ is the centre of $\mathbf{G}$, and $R(\mathbf{G})$ is a torus.
(3) $R(\mathbf{G}) \cap \mathbf{G}_{der}$ is finite, and $\mathbf{G} = R(\mathbf{G})\mathbf{G}_{der}$.

For the rest of this section, assume that $\mathbf{G}$ is a connected reductive group. Let $\mathbf{T}$ be a torus in $\mathbf{G}$. Then $Z_{\mathbf{G}}(\mathbf{T})$ is reductive. This fact is useful for inductive arguments. Now assume that $\mathbf{T}$ is maximal. Let $\mathfrak{t}$ be the Lie algebra of $\mathbf{T}$ and let $\Phi = \Phi(\mathbf{G}, \mathbf{T})$. Then

(1) $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ and $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$.
(2) If $\alpha \in \Phi$, let $\mathbf{T}_\alpha = (\operatorname{Ker}\alpha)^0$. Then $\mathbf{T}_\alpha$ is a torus, of codimension one in $\mathbf{T}$.
(3) If $\alpha \in \Phi$, let $\mathbf{Z}_\alpha = Z_{\mathbf{G}}(\mathbf{T}_\alpha)$. Then $\mathbf{Z}_\alpha$ is a reductive group of semisimple rank 1, and the Lie algebra $\mathfrak{z}_\alpha$ of $\mathbf{Z}_\alpha$ satisfies $\mathfrak{z}_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. The group $\mathbf{G}$ is generated by the subgroups $\mathbf{Z}_\alpha$, $\alpha \in \Phi$.
(4) The centre $Z(\mathbf{G})$ of $\mathbf{G}$ is equal to $\cap_{\alpha\in\Phi}\mathbf{T}_\alpha$.
(5) If $\alpha \in \Phi$, there exists a unique connected $\mathbf{T}$-stable (relative to conjugation by $\mathbf{T}$) subgroup $\mathbf{U}_\alpha$ of $\mathbf{G}$ having Lie algebra $\mathfrak{g}_\alpha$. Also, $\mathbf{U}_\alpha \subset \mathbf{Z}_\alpha$.
(6) Let $n \in N_{\mathbf{G}}(\mathbf{T})$, and let w be the corresponding element of $W = W(\mathbf{G}, \mathbf{T})$. Then $n\mathbf{U}_\alpha n^{-1} = \mathbf{U}_{w(\alpha)}$ for all $\alpha \in \Phi$.
(7) Let $\alpha \in \Phi$. Then there exists an isomorphism $\varepsilon_\alpha : \mathbf{G}_a \to \mathbf{U}_\alpha$ such that $t\,\varepsilon_\alpha(x)t^{-1} = \varepsilon_\alpha(\alpha(t)x)$, $t \in \mathbf{T}$, $x \in \mathbf{G}_a$.
(8) The groups $\mathbf{U}_\alpha$, $\alpha \in \Phi$, together with $\mathbf{T}$, generate the group $\mathbf{G}$.

Let $\langle\Phi\rangle$ be the subgroup of $X(\mathbf{T})$ generated by Φ and let $V = \langle\Phi\rangle \otimes_{\mathbb{Z}} \mathbb{R}$. Then the set Φ is a subset of the vector space V and is a root system. In general an *abstract root system* in a finite dimensional real vector space V, is a subset Φ of V that satisfies the following axioms:

(R1): Φ is finite, Φ spans V, and $0 \notin \Phi$.
(R2): If $\alpha \in \Phi$, there exists a reflection s_α relative to α such that $s_\alpha(\Phi) \subset \Phi$. (A *reflection* relative to α is a linear transformation sending α to $-\alpha$ that restricts to the identity map on a subspace of codimension one).
(R3): If α, $\beta \in \Phi$, then $s_\alpha(\beta) - \beta$ is an integer multiple of α.

A root system is *reduced* if it has the property that if $\alpha \in \Phi$, then $\pm\alpha$ are the only multiples of α which belong to Φ.

The *rank* of Φ is defined to be $\dim V$. The *abstract Weyl group* $W(\Phi)$ is the subgroup of $\mathbf{GL}(V)$ generated by the set $\{ s_\alpha \mid \alpha \in \Phi \}$.

If $\mathbf{T}$ is a maximal torus in $\mathbf{G}$, then $\Phi = \Phi(\mathbf{G}, \mathbf{T})$ is a root system in $V = \langle\Phi\rangle \otimes_{\mathbb{Z}} \mathbb{R}$, and it is reduced. The rank of Φ is equal to the semisimple rank of $\mathbf{G}$, and the abstract Weyl group $W(\Phi)$ is isomorphic to $W = W(\mathbf{G}, \mathbf{T})$.

A *base* of Φ is a subset $\Delta = \{ \alpha_1, \ldots, \alpha_\ell\}$, $\ell = \operatorname{rank}(\Phi)$, such that Δ is a basis of V and each $\alpha \in \Phi$ is uniquely expressed in the form $\alpha = \sum_{i=1}^{\ell} c_i\alpha_i$, where the c_i's are all integers, no two of which have different signs. The elements of Δ are called *simple roots*. The set of *positive roots* Φ^+ is the set of $\alpha \in \Phi$ such that the coefficients of the simple roots in the expression for α, as a linear combination of simple roots, are all nonnegative. Similarly, Φ^- consists of those $\alpha \in \Phi$ such that the coefficients are all nonpositive. Clearly Φ is the disjoint union of Φ^+ and Φ^-. Given $\alpha \in \Phi$, there exists a base containing α. Given a base Δ, the set $\{ s_\alpha \mid \alpha \in \Delta \}$ generates $W = W(\Phi)$. The subgroups $\mathbf{Z}_\alpha$, $\alpha \in \Delta$, generate $\mathbf{G}$. Equivalently, the subgroups $\mathbf{U}_\alpha$, $\alpha \in \Delta$, and $\mathbf{T}$, generate $\mathbf{G}$.

There is an inner product $(\cdot, \cdot)$ on V with respect to which each $w \in W$ is an orthogonal linear transformation. If α, $\beta \in \Phi$, then $s_\alpha(\beta) = \beta - (2(\beta, \alpha)/(\alpha, \alpha))\alpha$. A *Weyl chamber* in V is a connected component in the complement of the union of the hyperplanes orthogonal to the roots. The set of Weyl chambers in V and the set of bases of Φ correspond in a natural way, and W permutes each of them simply transitively.

If $\alpha \in \Phi$, there exists a unique $\alpha^\vee \in Y(\mathbf{T})$ such that $\langle \beta, \alpha^\vee \rangle = 2(\beta,\alpha)/(\alpha,\alpha)$ for all $\beta \in \Phi$. The set $\Phi^\vee$ of elements $\alpha^\vee$ (called co-roots) forms a root system in $\langle \Phi^\vee \rangle \otimes_{\mathbb{Z}} \mathbb{R}$, called the *dual* of Φ. The Weyl group $W(\Phi^\vee)$ is isomorphic to $W(\Phi)$, via the map $s_\alpha \mapsto s_{\alpha^\vee}$.

A root system Φ is said to be *irreducible* if Φ cannot be expressed as the union of two mutually orthogonal proper subsets. In general, Φ can be partitioned uniquely into a union of irreducible root systems in subspaces of V. The group $\mathbf{G}$ is *simple* (or *almost simple*) if $\mathbf{G}$ contains no proper nontrivial closed connected normal subgroup. When $\mathbf{G}$ is semisimple and connected, then $\mathbf{G}$ is simple if and only if Φ is irreducible.

The reduced irreducible root systems are those of type A_n, $n \geq 1$, B_n, $n \geq 1$, C_n, $n \geq 3$, D_n, $n \geq 4$, E_6, E_7, E_8, F_4, and G_2. For each $n \geq 1$ there is one irreducible nonreduced root system, BC_n. (These root systems are described in many of the references). If $n \geq 2$, the root system of $\mathbf{GL}_n(K)$ (relative to any maximal torus) is of type A_{n-1}. The root system of $\mathbf{Sp}_{2n}(K)$ is of type C_n, if $n \geq 3$, and of type A_1 and B_2 for $n = 1$ and 2, respectively.

The quadruple $\Psi(\mathbf{G},\mathbf{T}) = (X, Y, \Phi, \Phi^\vee) = (X(\mathbf{T}), Y(\mathbf{T}), \Phi(\mathbf{G},\mathbf{T}), \Phi^\vee(\mathbf{G},\mathbf{T}))$ is a *root datum*. An abstract root datum is a quadruple $\Psi = (X, Y, \Phi, \Phi^\vee)$, where X and Y are free abelian groups such that there exists a bilinear mapping $\langle \cdot,\cdot \rangle : X \times Y \to \mathbb{Z}$ inducing isomorphisms $X \simeq \mathrm{Hom}(Y,\mathbb{Z})$ and $Y \simeq \mathrm{Hom}(X,\mathbb{Z})$, and $\Phi \subset X$ and $\Phi^\vee \subset Y$ are finite subsets, and there exists a bijection $\alpha \mapsto \alpha^\vee$ of Φ onto $\Phi^\vee$. The following two axioms must be satisfied:

(RD1): $\langle \alpha, \alpha^\vee \rangle = 2$

(RD2): If $s_\alpha : X \to X$ and $s_{\alpha^\vee} : Y \to Y$ are defined by $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ and $s_{\alpha^\vee}(y) = y - \langle \alpha, y \rangle \alpha^\vee$, then $s_\alpha(\Phi) \subset \Phi$ and $s_{\alpha^\vee}(\Phi^\vee) \subset \Phi^\vee$ (for all $\alpha \in \Phi$).

The axiom **(RD2)** may be replaced by the equivalent axiom:

(RD2'): If $\alpha \in \Phi$, then $s_\alpha(\Phi) \subset \Phi$, and the s_α, $\alpha \in \Phi$, generate a finite group.

If $\Phi \neq \emptyset$, then Φ is a root system in $V := \langle \Phi \rangle \otimes_{\mathbb{Z}} \mathbb{R}$, where $\langle \Phi \rangle$ is the subgroup of X generated by Φ. The set $\Phi^\vee$ is the dual of the root system Φ.

The quadruple $\Psi^\vee = (Y, X, \Phi^\vee, \Phi)$ is also a root datum, called the *dual* of Ψ. A root datum is *reduced* if it satisfies a third axiom

(RD3): $\alpha \in \Phi \Longrightarrow 2\alpha \notin \Phi$.

The root datum $\Psi(\mathbf{G},\mathbf{T})$ is reduced.

An isomorphism of a root datum $\Psi = (X, Y, \Phi, \Phi^\vee)$ onto a root datum $\Psi' = (X', Y', \Phi', \Phi'^\vee)$ is a group isomorphism $f : X \to X'$ which induces a bijection of Φ onto Φ' and whose dual induces a bijection of $\Phi'^\vee$ onto $\Phi^\vee$. If $\mathbf{G}'$ is a linear algebraic group which is isomorphic to $\mathbf{G}$, and $\mathbf{T}'$ is a maximal torus in $\mathbf{G}'$, then the root data $\Psi(\mathbf{G},\mathbf{T})$ and $\Psi(\mathbf{G}',\mathbf{T}')$ are isomorphic.

If Ψ is a reduced root datum, there exists a connected reductive K-group $\mathbf{G}$ and a maximal torus $\mathbf{T}$ in $\mathbf{G}$ such that $\Psi = \Psi(\mathbf{G},\mathbf{T})$. The pair $(\mathbf{G},\mathbf{T})$ is unique up to isomorphism.

6. Parabolic subgroups

Let $\mathbf{G}$ be a connected linear algebraic group. The set of connected closed solvable subgroups of $\mathbf{G}$, ordered by inclusion, contains maximal elements. Such a maximal element is called a *Borel subgroup* of $\mathbf{G}$. If $\mathbf{B}$ is a Borel subgroup, then $\mathbf{G}/\mathbf{B}$ is a projective variety and any other Borel subgroup is conjugate to $\mathbf{B}$. If $\mathbf{P}$ is a closed subgroup of $\mathbf{G}$, then $\mathbf{G}/\mathbf{P}$ is a projective variety if and only if $\mathbf{P}$ contains a Borel subgroup. Such a subgroup is called a *parabolic subgroup*. If $\mathbf{P}$ is a parabolic subgroup, then $\mathbf{P}$ is connected and the normalizer $N_{\mathbf{G}}(\mathbf{P})$ of $\mathbf{P}$ in $\mathbf{G}$ is $\mathbf{P}$. If $\mathbf{P}$ and $\mathbf{P}'$ are parabolic subgroups containing a Borel subgroup $\mathbf{B}$, and $\mathbf{P}$ and $\mathbf{P}'$ are conjugate, then $\mathbf{P} = \mathbf{P}'$.

Now assume that $\mathbf{G}$ is a connected reductive linear algebraic group. Let $\mathbf{T}$ be a maximal torus in $\mathbf{G}$. Then $\mathbf{T}$ lies inside some Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. Let $\mathbf{U} = R_u(\mathbf{B})$ be the unipotent radical of $\mathbf{B}$. There exists a unique base Δ of $\Phi = \Phi(\mathbf{G}, \mathbf{T})$ such that $\mathbf{U}$ is generated by the groups $\mathbf{U}_\alpha$, $\alpha \in \Phi^+$, and $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$. Conversely if Δ is a base of Φ, then the group generated by $\mathbf{T}$ and by the groups $\mathbf{U}_\alpha$, $\alpha \in \Phi^+$, is a Borel subgroup of $\mathbf{G}$. Hence the set of Borel subgroups of $\mathbf{G}$ which contain $\mathbf{T}$ is in one to one correspondence with the set of bases of Φ. The Weyl group W permutes the set of Borel subgroups containing $\mathbf{T}$ simply transitively. The set of Borel subgroups containing $\mathbf{T}$ generates $\mathbf{G}$.

The Bruhat decomposition. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$, and let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ contained in $\mathbf{B}$. Then $\mathbf{G}$ is the disjoint union of the double cosets $\mathbf{B}w\mathbf{B}$, as w ranges over a set of representatives in $N_{\mathbf{G}}(\mathbf{T})$ of the Weyl group W ($BwB = Bw'B$ if and only if $w = w'$ in W).

Let $\mathbf{G}$, $\mathbf{B}$ and $\mathbf{T}$ be as above. Let Δ be the base of $\Phi(\mathbf{G}, \mathbf{T})$ corresponding to $\mathbf{B}$. If I is a subset of Δ, let W_I be the subgroup of W generated by the subset $S_I = \{\, s_\alpha \mid \alpha \in I \,\}$ of I. Let $\mathbf{P}_I = \mathbf{B}W_I\mathbf{B}$ (note that $\mathbf{P}_\emptyset = \mathbf{B}$). Then $\mathbf{P}_I$ is a parabolic subgroup of $\mathbf{G}$ (containing $\mathbf{B}$). A subgroup of $\mathbf{G}$ containing $\mathbf{B}$ is equal to $\mathbf{P}_I$ for some subset I of Δ. If I and J are subsets of Δ then $W_I \subset W_J$ implies $I \subset J$ and $\mathbf{P}_I \subset \mathbf{P}_J$ implies $I \subset J$. Also, $\mathbf{P}_I$ is conjugate to $\mathbf{P}_J$ if and only if $I = J$. A parabolic subgroup is called *standard* if it contains $\mathbf{B}$. Any parabolic subgroup $\mathbf{P}$ is conjugate to some standard parabolic subgroup.

Let $I \subset \Delta$. The set Φ_I of $\alpha \in \Phi$ such that α is an integral linear combination of elements of I forms a root system, with Weyl group W_I. The set of roots $\Phi(\mathbf{P}_I, \mathbf{T})$ of $\mathbf{P}_I$ relative to $\mathbf{T}$ is equal to $\Phi^+ \cup (\Phi^- \cap \Phi_I)$. Let $\mathbf{N}_I = R_u(\mathbf{P}_I)$. Then $\mathbf{N}_I$ is a $\mathbf{T}$-stable subgroup of $\mathbf{U} = \mathbf{B}_u$, and is generated by those $\mathbf{U}_\alpha$ which are contained in $\mathbf{N}_I$, that is, by those $\mathbf{U}_\alpha$ such that $\alpha \in \Phi^+$ and $\alpha \notin \Phi_I$. Let $\mathbf{T}_I = (\cap_{\alpha \in I} \mathrm{Ker}\,\alpha)^0$, and let $\mathbf{M}_I = Z_{\mathbf{G}}(\mathbf{T}_I)$. The set Φ_I coincides with the set of roots in Φ which are trivial on $\mathbf{T}_I$. The group $\mathbf{M}_I$ is reductive and is generated by $\mathbf{T}$ and by the set of $\mathbf{U}_\alpha$, $\alpha \in \Phi_I$, $\mathbf{T}_I$ is the identity component of the centre of $\mathbf{M}_I$, and $\Phi(\mathbf{M}_I, \mathbf{T}) = \Phi_I$. The Lie algebra of $\mathbf{M}_I$ is equal to $\mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_I} \mathfrak{g}_\alpha$ (here $\mathfrak{t}$ is the Lie algebra of $\mathbf{T}$). The group $\mathbf{M}_I$ normalizes $\mathbf{N}_I$ and $\mathbf{P}_I = \mathbf{M}_I \ltimes \mathbf{N}_I$. A *Levi factor* (or *Levi component*) of $\mathbf{P}_I$ is a reductive group $\mathbf{M}$ such that $\mathbf{P}_I = \mathbf{M} \ltimes \mathbf{N}_I$, and the decomposition $\mathbf{P}_I = \mathbf{M} \ltimes \mathbf{N}_I$ is called a *Levi decomposition* of $\mathbf{P}_I$. If $\mathbf{M}$ is a Levi factor of $\mathbf{P}_I$, then there exists $n \in \mathbf{N}_I$ such that $\mathbf{M} = n\mathbf{M}_I n^{-1}$. It is possible for $\mathbf{M}_I$ and $\mathbf{M}_J$ to be conjugate for distinct subsets I and J of Δ. More generally, if $\mathbf{P}$ is any parabolic subgroup of $\mathbf{G}$, $\mathbf{P}$ has Levi decompositions (which we can obtain via conjugation from Levi decompositions of a standard parabolic subgroup to which $\mathbf{P}$ is conjugate).

Note that if $\mathbf{P}$ is a proper parabolic subgroup of $\mathbf{G}$, then the semisimple rank of a Levi factor of $\mathbf{P}$ is strictly less than the semisimple rank of $\mathbf{G}$. This fact is often used in inductive arguments.

7. Reductive groups - relative theory

Let k be a subfield of K. Throughout this section, we assume that $\mathbf{G}$ is a connected reductive k-group. Then $\mathbf{G}$ has a maximal torus which is defined over k. We say that $\mathbf{G}$ is *k-split* if $\mathbf{G}$ has a maximal torus $\mathbf{T}$ which is k-split. If $\mathbf{G}$ is k-split and $\mathbf{T}$ is such a torus, then each $\mathbf{U}_\alpha$, $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$, is defined over k, and the associated isomorphism $\varepsilon_\alpha : \mathbf{G}_a \to \mathbf{U}_\alpha$ can be taken to be defined over k. If $\mathbf{G}$ contains no k-split tori, then $\mathbf{G}$ is said to be *k-anisotropic*. There exists a finite separable extension of k over which $\mathbf{G}$ splits.

Suppose that $\mathbf{G}$ and $\mathbf{G}'$ are connected reductive k-split k-groups which are isomorphic. Then $\mathbf{G}$ and $\mathbf{G}'$ are k-isomorphic.

The centralizer $Z_{\mathbf{G}}(\mathbf{T})$ of a k-torus $\mathbf{T}$ in $\mathbf{G}$ is reductive and defined over k, and if $\mathbf{T}$ is k-split, $Z_{\mathbf{G}}(\mathbf{T})$ is the Levi factor of a parabolic k-subgroup of $\mathbf{G}$. (Here, we say a closed subgroup $\mathbf{H}$ of $\mathbf{G}$ is a k-subgroup of $\mathbf{G}$ if $\mathbf{H}$ is a k-group). Any k-torus in $\mathbf{G}$ is contained in some maximal torus which is defined over k. If k is infinite, then $\mathbf{G}(k)$ is Zariski dense in $\mathbf{G}$.

The maximal k-split tori of $\mathbf{G}$ are all conjugate under $\mathbf{G}(k)$. Let $\mathbf{S}$ be a maximal k-split torus in $\mathbf{G}$. The *k-rank* of $\mathbf{G}$ is the dimension of $\mathbf{S}$. The *semisimple k-rank* of $\mathbf{G}$ is the k-rank of $\mathbf{G}/R(\mathbf{G})$. The finite group ${}_kW = N_{\mathbf{G}}(\mathbf{S})/Z_{\mathbf{G}}(\mathbf{S})$ is called the *k-Weyl group*. The set ${}_k\Phi = \Phi(\mathbf{G}, \mathbf{S})$ of roots of $\mathbf{G}$ relative to $\mathbf{S}$ is called the *k-roots* of $\mathbf{G}$. The k-roots form an abstract root system, which is not necessarily reduced, with Weyl group isomorphic to ${}_kW$. The rank of ${}_k\Phi$ is equal to the semisimple k-rank of $\mathbf{G}$.

A Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ might not be defined over k. We say that $\mathbf{G}$ is *k-quasisplit* if $\mathbf{G}$ has a Borel subgroup that is defined over k. If $\mathbf{P}$ is a parabolic k-subgroup of $\mathbf{G}$, then $R_u(\mathbf{P})$ is defined over k. A Levi factor $\mathbf{M}$ of a parabolic k-subgroup is called a Levi k-factor of $\mathbf{P}$ if $\mathbf{M}$ is a k-group. Any two Levi k-factors of $\mathbf{P}$ are conjugate by a unique element of $R_u(\mathbf{P})(k)$. If two parabolic k-subgroups of $\mathbf{G}$ are conjugate by an element of $\mathbf{G}$ then they are conjugate by an element of $\mathbf{G}(k)$. The group $\mathbf{G}$ contains a proper parabolic k-subgroup if and only if $\mathbf{G}$ contains a noncentral k-split torus, that is, if the semisimple k-rank of $\mathbf{G}$ is positive. The results described in this section give no information in the case where $\mathbf{G}$ has semisimple k-rank zero.

Let $\mathbf{P}_0$ be a minimal element of the set of parabolic k-subgroups of $\mathbf{G}$ (such an element exists, since the set is nonempty, as it contains $\mathbf{G}$). Any minimal parabolic k-subgroup of $\mathbf{G}$ is conjugate to $\mathbf{P}_0$ by an element of $\mathbf{G}(k)$. The group $\mathbf{P}_0$ contains a maximal k-split torus $\mathbf{S}$ of $\mathbf{G}$, and $Z_{\mathbf{G}}(\mathbf{S})$ is a k-Levi factor of $\mathbf{P}_0$. The semisimple k-rank of $Z_{\mathbf{G}}(\mathbf{S})$ is zero. Because $N_{\mathbf{G}}(\mathbf{S}) = N_{\mathbf{G}}(\mathbf{S})(k) \cdot Z_{\mathbf{G}}(\mathbf{S})$, $\mathbf{G}(k)$ contains representatives for all elements of ${}_kW$. The group ${}_kW$ acts simply transitively on the set of minimal parabolic k-subgroups containing $Z_{\mathbf{G}}(\mathbf{S})$.

Let $\mathrm{Lie}(Z_{\mathbf{G}}(\mathbf{S}))$ be the Lie algebra of $Z_{\mathbf{G}}(\mathbf{S})$. Then

$$\mathfrak{g} = \mathrm{Lie}(Z_{\mathbf{G}}(\mathbf{S})) \oplus \bigoplus_{\alpha \in {}_k\Phi} \mathfrak{g}_\alpha.$$

If $\alpha \in {}_k\Phi$ and $2\alpha \notin {}_k\Phi$, then $\mathfrak{g}_\alpha$ is a subalgebra of $\mathfrak{g}$. If α and $2\alpha \in {}_k\Phi$, then $\mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ is a subalgebra of $\mathfrak{g}$. For each $\alpha \in {}_k\Phi$, set

$$\mathfrak{g}_{(\alpha)} = \begin{cases} \mathfrak{g}_\alpha, & \text{if } 2\alpha \notin {}_k\Phi \\ \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}, & \text{if } 2\alpha \in {}_k\Phi. \end{cases}$$

There exists a unique closed connected unipotent k-subgroup $\mathbf{U}_{(\alpha)}$ of $\mathbf{G}$ which is normalized by $Z_{\mathbf{G}}(\mathbf{S})$ and has Lie algebra $\mathfrak{g}_{(\alpha)}$.

Let $\mathbf{P}_0$ be as above. Then there exists a unique base ${}_k\Delta$ of ${}_k\Phi$ such that $R_u(\mathbf{P}_0)$ is generated by the groups $\mathbf{U}_{(\alpha)}$, $\alpha \in {}_k\Phi^+$. The set of standard parabolic k-subgroups of $\mathbf{G}$ corresponds bijectively with the set of subsets of ${}_k\Phi$. Fix $I \subset {}_k\Delta$. Let $\mathbf{S}_I = (\cap_{\alpha \in I} \cap \operatorname{Ker} \alpha)^0$ and let ${}_k\Phi_I$ be the set of $\alpha \in {}_k\Phi$ which are integral linear combinations of the roots in I. Let ${}_kW_I$ be the subgroup of ${}_kW$ generated by the reflections s_α, $\alpha \in I$. The parabolic k-subgroup of $\mathbf{G}$ corresponding to I is $\mathbf{P}_I = \mathbf{P}_0 \cdot {}_kW_I \cdot \mathbf{P}_0$. The unipotent radical of $\mathbf{P}_I$ is equal to $\mathbf{N}_I$, the subgroup of $\mathbf{G}$ generated by the groups $\mathbf{U}_{(\alpha)}$, as α ranges over the elements of ${}_k\Phi^+$ which are not in ${}_k\Phi_I$. The k-subgroup $\mathbf{M}_I := Z_{\mathbf{G}}(\mathbf{S}_I)$ is a Levi k-factor of $\mathbf{P}_I$, $\Phi(\mathbf{M}_I, \mathbf{S}) = {}_k\Phi_I$, and ${}_kW_I = {}_kW(\mathbf{M}_I, \mathbf{S})$.

A parabolic k-subgroup of $\mathbf{G}$ is conjugate to exactly one $\mathbf{P}_I$, and it is conjugate to $\mathbf{P}_I$ by an element of $\mathbf{G}(k)$.

Relative Bruhat decomposition. Let $\mathbf{U}_0 = R_u(\mathbf{P}_0)$. Then $\mathbf{G}(k) = \mathbf{U}_0(k) \cdot N_{\mathbf{G}}(\mathbf{S})(k) \cdot \mathbf{U}_0(k)$, and $\mathbf{G}(k)$ is the disjoint union of the sets $\mathbf{P}_0(k)w\mathbf{P}_0(k)$, as w ranges over a set of representatives for elements of ${}_kW$ in $N_{\mathbf{G}}(\mathbf{S})(k)$.

A *parabolic subgroup* of $\mathbf{G}(k)$ is a subgroup of the form $\mathbf{P}(k)$, where $\mathbf{P}$ is a parabolic k-subgroup of $\mathbf{G}$. A subgroup of $\mathbf{G}(k)$ which contains $\mathbf{P}_0(k)$ is equal to $\mathbf{P}_I(k)$ for some $I \subset {}_k\Delta$. If $I \subset {}_k\Delta$, choosing representatives for ${}_kW_I$ in $N_{\mathbf{G}}(\mathbf{S})(k)$, we have $\mathbf{P}_I(k) = \mathbf{P}_0(k) \cdot {}_kW_I \cdot \mathbf{P}_0(k)$. The group $\mathbf{P}_I(k)$ is equal to its own normalizer in $\mathbf{G}(k)$. The Levi decomposition $\mathbf{P}_I = \mathbf{M}_I \ltimes \mathbf{N}_I$ carries over to the k-rational points: $\mathbf{P}_I(k) = \mathbf{M}_I(k) \ltimes \mathbf{N}_I(k)$. If I, $J \subset {}_k\Delta$ and $g \in \mathbf{G}(k)$, then $g\mathbf{P}_J(k)g^{-1} \subset \mathbf{P}_I(k)$ if and only if $J \subset I$ and $g \in \mathbf{P}_I(k)$.

8. Examples

EXAMPLE 8.1. $\mathbf{G} = \mathbf{GL}_n(K)$, $n \geq 2$.

The group $\mathbf{T} = \{\operatorname{diag}(t_1, t_2, \ldots, t_n) \mid t_i \in K^\times\}$ is a maximal torus in $\mathbf{G}$. For $1 \leq i \leq n$, let $\ell_i = (0, 0, \ldots, 0, 1, 0, \cdots, 0) \in \mathbf{Z}^n$, with the 1 occurring in the ith coordinate. The map $\sum_{i=1}^n k_i\ell_i \mapsto \chi_{\sum_{i=1}^n k_i\ell_i}$, where

$$\chi_{\sum_{i=1}^n k_i\ell_i}(\operatorname{diag}(t_1, \cdots, t_n)) = t_1^{k_1} \cdots t_n^{k_n},$$

is an isomorphism from $\mathbf{Z}^n$ to $X(\mathbf{T})$. If $\mu_{\sum_{i=1}^n k_i\ell_i}(t) = \operatorname{diag}(t^{k_1}, \cdots, t^{k_n})$, $t \in K^\times$, then the map $\sum_{i=1}^n k_i\ell_i \mapsto \mu_{\sum_{i=1}^n k_i\ell_i}$ is an isomorphism from $\mathbf{Z}^n$ to $Y(\mathbf{T})$. Also, $\langle \chi_{\Sigma k_i\ell_i}, \mu_{\Sigma \ell_i e_i}\rangle = \sum_{i=1}^n k_i\ell_i$. The root system $\Phi = \Phi(\mathbf{G}, \mathbf{T}) = \{\chi_{\ell_i - \ell_j} \mid 1 \leq i \neq j \leq n\}$.

For $1 \leq i \neq j \leq n$, let $E_{ij} \in M_n(K) = \mathfrak{g}$ be the matrix having a 1 in the ij^{th} entry, and zeros elsewhere. If $\alpha = \chi_{\ell_i - \ell_j}$, $i \neq j$, then $\mathfrak{g}_\alpha$ is spanned by E_{ij}, and

$\mathbf{U}_\alpha = \{ I_n + tE_{ij} \mid t \in K \}$. The reflection s_α permutes ℓ_i and ℓ_j, and fixes all ℓ_k with $k \notin \{i, j\}$. The co-root $\alpha^\vee$ is $\mu_{\ell_i - \ell_j}$ The Weyl group W is isomorphic to the symmetric group S_n. The root system $\Phi \simeq \Phi^\vee$ is of type A_{n-1}.

The set $\Delta := \{ \chi_{\ell_i - \ell_{i+1}} \mid 1 \leq i \leq n-1 \}$ is a base of Φ. The corresponding Borel subgroup $\mathbf{B}$ is the subgroup of $\mathbf{G}$ consisting of upper triangular matrices.

If $I \subset \Delta$, there exists a partition $(n_1, n_2, \cdots, n_r)$ of n (n_i a positive integer, $1 \leq i \leq r$, $n_1 + n_2 + \cdots + n_r = n$), such that

$$\mathbf{T}_I = \{ \operatorname{diag}(\underbrace{a_1, \ldots, a_1}_{n_1 \text{ times}}, \underbrace{a_2, \ldots, a_2}_{n_2 \text{ times}}, \ldots, \underbrace{a_r, \ldots, a_r}_{n_r \text{ times}}) \mid a_1, a_2, \ldots, a_r \in K^\times \}$$

The group $\mathbf{M}_I := \mathbf{Z_G}(\mathbf{T}_I)$ is isomorphic to $\mathbf{GL}_{n_1}(K) \times \mathbf{GL}_{n_2}(K) \times \cdots \times \mathbf{GL}_{n_r}(K)$, $\mathbf{N}_I$ consists of matrices of the form

$$\begin{bmatrix} I_{n_1} & * & * & * \\ & I_{n_2} & * & \vdots \\ 0 & & \ddots & * \\ & & & I_{n_r} \end{bmatrix},$$

and $\mathbf{P}_I = \mathbf{M}_I \ltimes \mathbf{N}_I$.

Example 8.2. $\mathbf{G} = \mathbf{Sp}_4(K)$ (the 4×4 symplectic group). Let

$$J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

Then $\mathbf{G} = \{\mathfrak{g} \in \mathbf{GL}_4(K) \mid {}^t gJg = J\}$ and $\mathfrak{g} = \{X \in M_4(K) \mid {}^t XJ + JX = 0\}$.

The group $\mathbf{T} := \{ \operatorname{diag}(a, b, b^{-1}, a^{-1}) \mid a, b \in K^\times\}$ is a maximal torus in $\mathbf{G}$ and $X(\mathbf{T}) \simeq \mathbf{Z} \times \mathbf{Z}$, via $\chi_{(i,j)} \leftrightarrow (i, j)$, where $\chi_{(i,j)}(\operatorname{diag}(a, b, b^{-1}, a^{-1})) = a^i b^j$. And $Y(\mathbf{T}) \simeq \mathbf{Z} \times \mathbf{Z}$, via $\mu_{(i,j)} \leftrightarrow (i, j)$, where $\mu_{(i,j)}(t) = (\operatorname{diag}(t^i, t^j, t^{-j}, t^{-i})$. Note that $\langle \chi_{(i,j)}, \mu_{(k,\ell)} \rangle = ki + j\ell$.

Let $\alpha = \chi_{(1,-1)}$ and $\beta = \chi_{(0,2)}$. Then

$$\Phi = \{ \pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(2\alpha + \beta \},$$

$\Delta := \{\alpha, \beta\}$ is a base of $\Phi = \Phi(\mathbf{G}, \mathbf{T})$, and

$$\mathfrak{g}_\alpha = \operatorname{Span}_K(E_{12} - E_{34}), \quad \mathfrak{g}_{-\alpha} = \operatorname{Span}_K(E_{21} - E_{43}) \quad \mathfrak{g}_\beta = \operatorname{Span}_K E_{23}$$
$$\mathfrak{g}_{\alpha+\beta} = \operatorname{Span}_K(E_{13} + E_{24}), \quad \mathfrak{g}_{2\alpha+\beta} = \operatorname{Span}_K E_{14}, \quad \text{etc.}$$

Identifying α and β with $(1, -1)$ and $(0, 2) \in \mathbf{Z} \times \mathbf{Z}$, respectively, we have $s_\alpha(1, -1) = (-1, 1) = -\alpha$ and $s_\alpha(1, 1) = (1, 1)$. The corresponding element of $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ is represented by the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We also have $s_\beta(0,2) = (0,-2) = -\beta$ and $s_\beta(1,0) = (1,0)$. The corresponding element of $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ is represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Weyl group $W = W(\Phi)$ is equal to $\{\, 1,\ s_\alpha,\ s_\beta,\ s_\alpha s_\beta,\ s_\beta s_\alpha,\ s_\beta s_\alpha s_\beta,\ s_\alpha s_\beta s_\alpha,\ (s_\beta s_\alpha)^2 \,\}$ which is isomorphic to the dihedral group of order 8.

The dual root system $\Phi^\vee$ is described by

$$\begin{aligned} \Phi^\vee &= \{\pm\alpha^\vee,\ \pm\beta^\vee,\ \pm(\alpha+\beta)^\vee,\ \pm(2\alpha+\beta)^\vee\} \\ \alpha^\vee &= (1,\,-1) \quad (\alpha+\beta)^\vee = (1,\,1) \\ \beta^\vee &= (0,\,1) \quad (2\alpha+\beta)^\vee = (1,\,0) \end{aligned}$$

The root system Φ is of type C_2 and $\Phi^\vee$ is of type B_2, isomorphic to C_2.

Remark 8.3. If $n > 2$ the root system of $\mathbf{Sp}_{2n}(K)$ is of type C_n, and the dual is of type B_n, and B_n and C_n are not isomorphic.

The Borel subgroup of $\mathbf{G}$ which corresponds to Δ is the subgroup $\mathbf{B}$ of upper triangular matrices in $\mathbf{G}$. Apart from $\mathbf{G}$ and $\mathbf{B}$, there are two standard parabolic subgroups, $\mathbf{P}_\alpha$ and $\mathbf{P}_\beta$, attached to the subsets $\{\alpha\}$ and $\{\beta\}$ of Δ, respectively. It is easy to check that

$$\mathbf{T}_\alpha = (\mathrm{Ker}\ \alpha)^\circ = \{\mathrm{diag}\ (a,\ a,\ a^{-1}\,a^{-1}) \mid a \in K^\times\}$$

$$\mathbf{M}_\alpha = \mathbf{Z}_{\mathbf{G}}(\mathbf{T}_\alpha) = \left\{ \begin{bmatrix} A & 0 \\ 0 & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} {}^tA^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} \;\middle|\; A \in \mathbf{GL}_2(K) \right\}$$

$$\mathbf{N}_\alpha = \left\{ \begin{bmatrix} I_2 & B \\ 0 & I_2 \end{bmatrix} \;\middle|\; B \in M_2(K),\ {}^tB = B \right\}$$

$$\mathbf{T}_\beta = (\mathrm{Ker}\ \beta)^\circ = \{\,\mathrm{diag}\,(a,\ 1,\ 1,\ a^{-1}) \mid a \in K^\times\,\}$$

$$\mathbf{M}_\beta = \mathbf{Z}_{\mathbf{G}}(\mathbf{T}_\beta) = \left\{ \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & c_{11} & c_{12} & 0 \\ 0 & c_{21} & c_{22} & 0 \\ 0 & 0 & 0 & d^{-1} \end{bmatrix} \;\middle|\; d \in K^\times,\ c_{11}c_{22} - c_{12}c_{21} = 1 \right\} \simeq \mathbf{SL}_2(K) \times K^\times$$

$$\mathbf{N}_\beta = \left\{ \begin{bmatrix} 1 & x & y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{bmatrix} \;\middle|\; x, y, z \in K \right\}$$

9. Comments on references

For the basic theory of linear algebraic groups, see [**B1**], [**H**] and [**Sp2**], as well as the survey article [**B2**]. For information on reductive groups defined over non algebraically closed fields, the main reference is [**BoT1**] and [**BoT2**]. Some material appears in [**B1**], and there is a survey of rationality properties at the end of [**Sp2**]. See also the survey article [**Sp1**]. For the classification of semisimple algebraic groups, see [**Sa**] and [**T2**]. For information on reductive groups over local nonarchimedean fields, see [**BrT1**], [**BrT2**], and the article [**T1**]. Adeles and algebraic groups are discussed in [**W**].

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