<u>thm</u>: (Stone Weierstrauss) Fix $[a,b]$. Let $\mathcal{P} = \{$polynomials with rational coefficients$\}$. Then $\mathcal{P}$ is dense in $(C([a,b]), \|\cdot\|_\infty)$

<u>note</u>: Since $\mathcal{P}$ is countable, we have a countable dense set in $(C([a,b]), \|\cdot\|_\infty)$

<u>note</u>: $\mathcal{P}$ is <u>not</u> dense in $(C(\mathbb{R}), \|\cdot\|_\infty)$. If the interval isn't closed you're going to lose the countability of a dense subset

<u>thm</u>: Let $\tilde{\mathcal{P}} = \{f 1_{[-N,N]} \mid f \in \mathcal{P} \text{ and } N \in \mathbb{N}\}$. Then $\tilde{\mathcal{P}}$ is dense in $(L^2(\mathbb{R}), \|\cdot\|_2)$

<u>proof</u>: Let $f \in L^2(\mathbb{R})$. Fix $\varepsilon > 0$. Since $C_0(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, $\exists f_0 \in C_0(\mathbb{R})$ so that

$$\|f - f_0\|_2 < \varepsilon/2$$

It suffices to find $g \in \tilde{\mathcal{P}}$ and N so that

$$\|g 1_{[-N,N]} - f_0\|_2 < \varepsilon/2.$$

Since f_0 has compact support, $\exists N$ so that

$$\operatorname{supp}(f_0) \subseteq [-N, N].$$

By Stone-Weierstrauss, $\exists g \in \mathcal{P}$ so that

$$\max_{x \in [-N,N]} |f_0 - g(x)| = \| f_0 - g 1_{[-N,N]} \|_\infty < \frac{\varepsilon}{2\sqrt{2N}}$$

Now

$$\| g 1_{[-N,N]} - f_0 \|_2 = \sqrt{\int |g 1_{[-N,N]} - f_0|^2}$$

$$= \sqrt{\int_{[-N,N]} |g - f_0|^2}$$

$$\leq \| f_0 - g 1_{[-N,N]} \|_\infty \sqrt{\int_{[-N,N]} 1}$$

$$= \| f_0 - g 1_{[-N,N]} \|_\infty \sqrt{2N} < \frac{\varepsilon}{2}$$

as desired. //

<u>Note</u>: Convince yourself that $\tilde{\mathcal{P}}$ is dense in $L^p(\mathbb{R})$ for all p in $[1, \infty)$.

<u>Theorem</u>: Let $(X, (\cdot,\cdot))$ be a complete real inner product space

If X is separable (has a countable dense subset) then $\exists$ a countable orthonormal basis for X.

i.e. $\exists \{x_n\}_1^\infty$ so that $(x_m, x_n) = \delta_{mn}$ and given $y \in X$, $y = \sum_{n=1}^{\infty} (y, x_n) x_n$

Note: As a result, if $f \in L^2$ then $f = \sum_{n=1}^{\infty} \alpha_n \psi_n$ where $\{\psi_n\}$ orthonormal and $\alpha_k = (f, \psi_k)$. This is used in Quantum mechanics all the time

Note: In some books (e.g. Beals & Folland) a Hilbert Space is a complete inner product space. In other books (e.g. Kolmogorov + Fomin, all physics books) a Hilbert Space is a complete inner product space that is separable.

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§12E Differentiating the Integral.

From Calculus, you know that if f is continuous and $F(x) := \int_a^x f(t)\,dt$ then F is differentiable and $F'(x) = f(x)$ for all x.

From your HW (page 165, #4) you proved that if $f, g \in L^1(\mathbb{R})$ and $F(x) := \int_{[a,x]} f$, $G(x) := \int_{[a,x]} g$

then $\int_{[a,b]} Fg = F(b)G(b) - \int_{[a,b]} fG$. That is you proved an integration by parts rule holds. But you did this w/o proving $F'(x) = f(x)$ and $G'(x) = g(x)$!

Theorem: Let $f \in L^1(\mathbb{R})$. Fix $a \in \mathbb{R}$. Define

$$F(x) = \begin{cases} \int_{[a,x]} f & \text{if } x \geq a \\ -\int_{[x,a]} f & \text{if } x < a \end{cases}$$

Then F is continuous, and for almost all x $F'(x)$ exists and equals $f(x)$.

To prove this theorem, we will use the following:

Lemma (Hardy-Littlewood Inequality)
Assume $h \in L^1(\mathbb{R})$ and define

$$h^*(x) := \sup \left\{ \frac{1}{|I|} \int_I |h| \;\middle|\; \begin{array}{l} I \text{ is an open} \\ \text{interval, } x \in I \end{array} \right\}.$$

Then given $\lambda > 0$ $\quad m(\{h^* > \lambda\}) \leq \frac{5}{\lambda} \|h\|_1$.

h^* is called the Hardy-Littlewood Maximal function.
(Note: some books require that I has midpoint x.)

proof of theorem:

First we prove that $F \in C(\mathbb{R})$. Assume $x_n \to x$. We show that $F(x_n) \to F(x)$.

case 1: $x > a$. Then for n sufficiently large, $x_n > a$.

$$|F(x_n) - F(x)| = \left| \int_{[a,x_n]} f - \int_{[a,x]} f \right|$$

$$= \left| \int_{E_n} f \right| \quad \text{where } E = [x_n, x] \text{ or } [x, x_n]$$

$$= \left| \int f 1_{E_n} \right|$$

let $g_n := f 1_{E_n}$. Note that $g_n \to f 1_{\{x\}}$ pointwise. Also, $|g_n| \le |f| \in L^1(\mathbb{R})$ for all n. Hence, by Lebesgue Dominated Convergence,

$$\int \lim_{n\to\infty} g_n = \lim_{n\to\infty} \int g_n$$

$$\|$$

$$\int f 1_{\{x\}} = 0.$$

Thus $\lim_{n\to\infty} |F(x) - F(x_n)| = \left| \lim_{n\to\infty} \int g_n \right| = 0$

as desired.

Case 2: $x \le a$. In this case, choose $b < x$ and write

$$F(x) = \int_{[b,x]} f - \int_{[b,a]} f$$

now repeat the argument of case 1.

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we observe that our theorem holds for any $f \in C(\mathbb{R})$.

We want to leverage off of the density of $C_0(\mathbb{R})$ in $L^1(\mathbb{R})$ somehow. Since we know the theorem holds on this dense subset.

to be continued...