Mat 357 March 7, 2005

Proof 2 (of Cauchy-Schwarz)

We write $f = -\lambda g + h$ where $h \perp g$.

$$h = f - \frac{(f,g)}{(g,g)} g$$

First: is $h \perp g$? $(h,g) = \left(f - \frac{(f,g)}{(g,g)} g, g\right)$

$$= (f,g) - \frac{(f,g)}{(g,g)} (g,g) = 0 \checkmark$$

Second: $f = -\lambda g + h$?

If $\lambda = -\frac{(f,g)}{(g,g)}$ then $f = -\lambda g + h$ ✓

Now, $\|f\|^2 = (f,f) = (-\lambda g + h, -\lambda g + h)$

$$= \lambda^2 \|g\|^2 + \|h\|^2 \quad \text{since } h \perp g$$

$$\geq \lambda^2 \|g\|^2 \quad \text{since } \|h\|^2 \geq 0$$

thus, $\|f\|^2 \geq \left(-\frac{(f,g)}{(g,g)}\right)^2 \|g\|^2 = \frac{(f,g)^2}{\|g\|^2}$ and

So $|(f,g)| \leq \|f\| \, \|g\|$. //

Note : The two proofs are closely related. In the first proof, you allow $\lambda$ to have any value and minimize $\phi$. The thing you're minimizing, $\phi(\lambda) = \|h_\lambda\|$ where $h_\lambda = f + \lambda g$. This is minimized when $h_\lambda \perp g$. Which happens at a specific value of $\lambda$. i.e. $\lambda_{min}$.

Note: The second proof gives the argument for

claim: $|(f,g)| = \|f\| \, \|g\| \iff f = \lambda g$ some $\lambda$.

Proof:

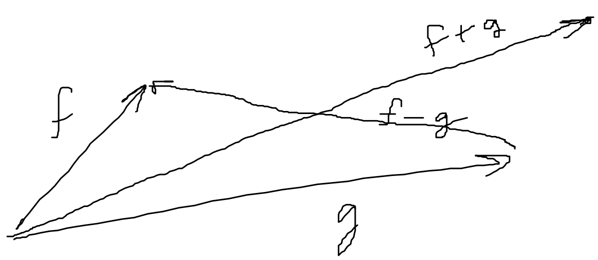
($\Leftarrow$) Assume $f = \lambda g$. Then $|(f,g)| = |\lambda| \, \|g\|^2$
$= \|g\| \, \|f\|$ ✓

($\Rightarrow$) Assume $f \notin \operatorname{span}\{g\}$. Then $f = -\lambda g + h$ for some $h \neq 0$ where $h \perp g$. Hence $\|f\|^2 = \lambda^2 \|g\|^2 + \|h\|^2 > \lambda^2 \|g\|^2$ since $h \neq 0$. Since $\lambda = -\dfrac{(f,g)}{(g,g)}$ it follows that $|(f,g)| < \|f\| \, \|g\|$. //

Under what conditions can a normed vector space have its norm be one induced by an inner product?

Theorem: If $(X, \|\cdot\|)$ is a normed vector space then $\|\cdot\|$ is induced by an inner product iff
$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2)$$
for all $f, g \in X$.

Geometrically, this is called the parallelogram law because of its relation to the sides & diagonals of the parallelogram




Recall $(\mathbb{R}^n, \|\cdot\|_p)$ where $\|\vec{x}\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$

then this norm ~~lbrg~~ is induced by an inner product $\Leftrightarrow p=2$. Why? take

$f = (1,1,0,0,\dots 0) \quad g = (1,-1,0,\dots 0)$

$f+g = (2,0,\dots 0) \quad f-g = (0,2,0\dots 0)$

Then $\|f\|_p = \|g\|_p = 2^{1/p}$ and $\|f+g\|_p = \|f-g\|_p = 2$

Hence there's an inducing inner product iff

$$2^2 + 2^2 = 2\left(2^{2/p} + 2^{2/p}\right) \iff p = 2$$

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This example allows us to ask the same question about $(L^p(\mathbb{R}), \|\cdot\|_p)$ where

$$\|f\|_p = \sqrt[p]{\int |f|^p}$$

take $f = \mathbb{1}_{[0,1]} + \mathbb{1}_{[2,3]}$ $\quad g = \mathbb{1}_{[0,1]} - \mathbb{1}_{[2,3]}$

then $f+g = 2 \cdot \mathbb{1}_{[0,1]}$ and $f-g = 2\,\mathbb{1}_{[2,3]}$

$\Rightarrow$ $\|f\|_p = \|g\|_p = 2^{1/p}$ and $\|f+g\|_p = \|f-g\|_p = 2$

and $\|\cdot\|_p$ is induced by an inner product iff $p=2$.

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What about $(C([a,b]), \|\cdot\|_\infty)$? Take $[a,b] = [0, \pi/2]$ and $f(t) = \cos(t)$, $g(t) = \sin(t)$. Then $\|f\|_\infty = \|g\|_\infty = 1$, $\|f+g\|_\infty = \sqrt{2}$ and $\|f-g\|_\infty = 1$. So the $\|\cdot\|_\infty$ norm cannot come from an inner product.

For other $[a,b]$, just dilate/compress $f$ and $g$ as needed.

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As you know from linear algebra, inner products allow ~~ses~~ us to talk about "perpendicular" and given a subspace $Y$ one can construct $Y^\perp$ and decompose a space $X$ into orthogonal subspaces and all sorts of cool things.

proof of theorem:

($\Leftarrow$) Assume $\|\cdot\|$ is induced by the inner product $(\cdot,\cdot)$. Then we need to show the parallelogram law holds. i.e. we need to show that

$$(f+g, f+g) = 2\left[(f,f) + (g,g)\right] + (f-g, f-g)$$

$$\text{LHS} = (f,f) + 2(f,g) + (g,g) + (f,f) - 2(f,g) + (g,g) = \text{RHS} \checkmark$$

($\Rightarrow$) assume the norm satisfies the parallelogram law. We will now define an "inner product" and prove it satisfies all the requirements.

$$(f,g) := \frac{1}{4}\left[\|f+g\|^2 - \|f-g\|^2\right]$$

1) $(f,f) = \frac{1}{4}\left[\|f+f\|^2 - \|f-f\|^2\right] = \|f\|^2 \geq 0$ ✓

$(f,f) = 0 \iff \|f\| = 0 \iff f = 0$ ✓

2) $(f,g) = \frac{1}{4}\left[\|f+g\|^2 - \|f-g\|^2\right] = \frac{1}{4}\left[\|g+f\|^2 - \|g-f\|^2\right]$

$= (g,f)$ ✓

3) Showing $(f+g,h) = (f,h) + (g,h)$ will take some sweat. Here goes! Let

$$\phi(f,g,h) := 4\left[(f+g,h) - (f,h) - (g,h)\right]$$

It suffices to show $\phi \equiv 0 \ \forall f,g,h$

$$\phi(f,g,h) = \|f+g+h\|^2 - \|f+g-h\|^2 - \|f+h\|^2 + \|f-h\|^2 \quad (*)$$
$$- \|g+h\|^2 + \|g-h\|^2$$

by definition of $(\cdot,\cdot)$

From the parallelogram laws, we know

$$\|f \pm h + g\|^2 = 2\|f \pm h\|^2 + 2\|g\|^2 - \|f \pm h - g\|^2$$

Substituting this in for the first 2 terms...

$$\phi(f,g,h) = \|f+h\|^2 - \|f-h\|^2 - \|f+h-g\|^2 + \|f-h-g\|^2 - \|g+h\|^2 + \|g-h\|^2 \tag{**}$$

Adding $(*)$ and $(**)$ and then dividing by 2,

$$\phi(f,g,h) = -\|g+h\|^2 + \|g-h\|^2 + \frac{1}{2}\left(\|f+g+h\|^2 + \|g+h-f\|^2\right) - \frac{1}{2}\left(\|g-h+f\|^2 + \|g-h-f\|^2\right)$$

Applying the parallelogram law to the last two factors,

$$\phi(f,g,h) = -\|g+h\|^2 + \|g-h\|^2 + \left(\|g+h\|^2 + \|f\|^2\right) - \left(\|g-h\|^2 + \|f\|^2\right)$$
$$= 0.$$

This proves $\phi(f,g,h) = 0$ as desired.

4) Showing $(cf,g) = c(f,g)$ $\forall c \in \mathbb{R}$. Note that at this point we can use properties 1), 2), and 3) for inner products if we need to.

Let $\phi(c) := (cf, g) - c(f, g)$

<u>obs 1</u>: $\phi(0) = (0f, g) - 0(f, g)$

$$= (0, g) = \frac{1}{4}\left[\|0+g\|^2 - \|0-g\|^2\right] = 0.$$

<u>obs 2</u>: $\phi(-1) = (-f, g) + (f, g)$

$$= \frac{1}{4}\left[\|-f+g\|^2 - \|-f-g\|^2\right]$$

$$+ \frac{1}{4}\left[\|f+g\|^2 - \|f-g\|^2\right] = 0$$

hence $(-f, g) = -(f, g) \ \forall f, g$.

Now let $n \in \mathbb{Z}$ then

$$(nf, g) = (\operatorname{sgn}(n)[\underbrace{f+f+\dots+f}_{|n| \text{ times}}], g)$$

$$= \operatorname{sgn}(n)\,(\underbrace{f+f+\dots+f}_{|n| \text{ times}}, g) \quad \text{by obs 2}$$

$$= \operatorname{sgn}(n)[\underbrace{(f, g) + (f, g) + \dots + (f, g)}_{|n| \text{ times}}] \quad \text{by 3)}$$

$$= |n| \operatorname{sgn}(n)\,(f, g) = n(f, g)$$

hence $\phi(n) = 0 \quad \forall n \in \mathbb{Z}$.

Now let $p, q \in \mathbb{Z}$ then

$$\left(\frac{p}{q}f, g\right) = p\left(\frac{1}{q}f, g\right)$$

since we've shown $(nf, g) = n(f, g)$ $\forall n \in \mathbb{Z}$

$$= \frac{p}{q}\, q\left(\frac{1}{q}f, g\right)$$

$$= \frac{p}{q}\left(q\frac{1}{q}f, g\right) \quad \text{"} \quad \text{"}$$

$$= \frac{p}{q}(f, g)$$

This shows that $\phi(c) = 0$ $\forall c \in \mathbb{Q}$.

If I can show that $\phi : \mathbb{R} \to \mathbb{R}$ is continuous, then we're done by the density of $\mathbb{Q}$ in $\mathbb{R}$. Since then $\phi(c) = 0$ $\forall c \in \mathbb{R}$, as desired.

$$|\phi(c_0) - \phi(c_1)| = |(c_0 f, g) - c_0(f, g) - (c_1 f, g) + c_1(f, g)|$$

$$\leq |(c_0 f, g) - (c_1 f, g)| + |(-c_0 + c_1)(f, g)|$$

$$= |((c_0 - c_1)f, g)| + |c_0 - c_1|\,|(f, g)| \quad \text{by 3)}$$

$$= \frac{1}{4}\left|\,\|(c_0 - c_1)f + g\|^2 - \|(c_0 - c_1)f - g\|^2\right| + |c_0 - c_1|\,|(f, g)|$$

$$= \frac{1}{4} \Big| \|(c_0 - c_1)f + g\| - \|(c_0 - c_1)f - g\| \Big| \times$$

$$\Big| \|(c_0 - c_1)f + g\| + \|(c_0 - c_1)f - g\| \Big|$$

$$+ |c_0 - c_1| \, |(f, g)|$$

$$\leq \frac{1}{2} |c_0 - c_1| \|f\| \times \Big| \|(c_0 - c_1)f + g\| + \|(c_0 - c_1)f - g\| \Big|$$

$$+ |c_0 - c_1| \, |(f, g)|$$

here I used the triangle inequality to find

$$\Big| \|g + (\ )\| - \|g - (\ )\| \Big| \leq 2\|(\ )\|$$

$$= 2|c_0 - c_1| \|f\|$$

$$\leq |c_0 - c_1| \, M$$

where $M$ depends on $\|f\|$, $\|g\|$, $\|f+g\|$ and $\|f-g\|$.

This shows that $\phi : \mathbb{R} \to \mathbb{R}$ is continuous, as claimed. //

Finally, $(L^2(\mathbb{R}), \|\cdot\|_2)$ is complete and

$$\overline{(C_0(\mathbb{R}), \|\cdot\|_2)} = (L^2(\mathbb{R}), \|\cdot\|_2)$$

<u>proof 1</u> of the completeness of $L^p(\mathbb{R})$ for $1 \le p < \infty$:

If $p=1$ then the result was already proven in §12C. So assume $1 < p < \infty$.
By the construction in the book, given a cauchy sequence $\{f_n\}_1^\infty$ in $L^p(\mathbb{R})$, we construct a subsequence $\{g_n\}_1^\infty$ so that $\|g_{k+1} - g_k\|_p < 1/2^k$ and $g_n \to f$ pointwise almost everywhere.
We now prove that $f \in L^p(\mathbb{R})$ and that $\|f_n - f\|_p \to 0$ as $n \to \infty$.
Fix $\varepsilon > 0$. $\exists N_\varepsilon$ so that $m, n \ge N_\varepsilon$ then $\|f_n - f_m\|_p < \varepsilon$. Specifically, $\|g_k - f_m\|_p < \varepsilon$ if $m \ge N_\varepsilon$ and $N(k) \ge N_\varepsilon$. Fix $m \ge N_\varepsilon$. Then by Fatou's lemma,

$$\int \varliminf_{k \to \infty} |f_{N(k)} - f_m|^p \le \varliminf_{k \to \infty} \int |f_{N(k)} - f_m|^p < \varepsilon^p$$

and since $f_{N(k)} = g_k \to f$ almost everywhere,

$$\int |f - f_m|^p = \int \varliminf_{k \to \infty} |f_{N(k)} - f_m|^p$$

proving $\int |f - f_m|^p < \varepsilon^p$. This proves that $f - f_m \in L^p(\mathbb{R})$. Since $f_m \in L^p(\mathbb{R})$ and $L^p(\mathbb{R})$ is a vector space we conclude $f \in L^p(\mathbb{R})$. Note that we've also shown that for any $m \ge N_\varepsilon$, $\|f - f_m\|_p < \varepsilon$, proving $f_m \to f$ in $L^p(\mathbb{R})$. //

Lemma: Assume $\frac{1}{p}+\frac{1}{q}=1$ and $1<p<\infty$. Assume $g$ is a measurable function and that $fg\in L^1(\mathbb{R})$ for all ISF $f$. Let

$$M_q(g) := \sup\left\{ \left|\int fg\right| \;\middle|\; f \text{ an ISF and } \|f\|_p = 1\right\}$$

If $M_q(g)<\infty$ then $g\in L^q(\mathbb{R})$ and $\|g\|_q = M_q(g)$.

proof 2 of the completeness of $L^p(\mathbb{R})$ for $1\le p<\infty$

As in proof 1, it suffices to prove that $f\in L^p(\mathbb{R})$ and that $\|f_n - f\|_p \to 0$ as $n\to\infty$.

Let $h\in L^q(\mathbb{R})$ Then

$$(f(x)-g_n(x))h(x) = \sum_{k=n}^{\infty} (g_{k+1}(x)-g_n(x))h(x)$$

$$\Rightarrow \left|\int (f-g_n)h\right| \le \int |f-g_n||h| \le \sum_{k=n} \int |g_{k+1}-g_n||h|$$

$$\le \sum_{k=n}^{\infty} \|h\|_q \|g_{k+1}-g_k\|_p < 2^{1-n}\|h\|_q$$

Hence for all $h\in L^q(\mathbb{R})$, $(f-g_n)h$ is integrable and $\exists\, C<\infty$ so that

$$|(f-g_n, h)| \le C\|h\|_q.$$

By the lemma, $C \ge M_p(f-g_n) = \|f-g_n\|_p$

and so $\|f-g_n\|_p < \frac{1}{2^n}$ $\quad f-g_n \in L^p(\mathbb{R}) \Rightarrow f\in L^p(\mathbb{R})$

since $g_n\to f$ in $L^p(\mathbb{R})$ and $\{f_n\}$ Cauchy in $L^p(\mathbb{R})$

$\Rightarrow f_n \to f$ in $L^p(\mathbb{R})$. //

proof of lemma:

First, note that if $f$ is a bounded measurable function that vanishes outside a set of finite measure and $\|f\|_p = 1$, then $|\int fg| \le M_q(g)$. Why? $\exists$ a sequence of ISF so that $|f_n| \le |f|$ and $f_n \to f$ in $L^p(\mathbb{R})$. Further,

$$|f_n| \le \|f\|_\infty 1_E$$

where $E = \{f \neq 0\}$ and by assumption, $g 1_E \in L^1(\mathbb{R})$ we can apply the Lebesgue Dominated Convergence theorem to the sequence $\{f_n g 1_E\}$. It converges pointwise a.e to $f g 1_E = fg$ and $|f_n g 1_E| \le \|f\|_\infty |g 1_E| \in L^1$

Hence

$$\int fg = \int \lim_{n\to\infty} f_n g 1_E = \lim_{n\to\infty} \int f_n g 1_E = \lim_{n\to\infty} \|f_n\|_p \int \frac{f_n}{\|f_n\|_p} 1_E$$

So
$$\left|\int fg\right| = \lim_{n\to\infty} \|f_n 1_E\|_p \left|\int \frac{f_n 1_E}{\|f_n 1_E\|_p} g\right|$$

$$\le \lim_{n\to\infty} \|f_n 1_E\|_p M_q(g)$$

since $\dfrac{f_n 1_E}{\|f_n 1_E\|_p}$ is an ISF w/ $L^p$ norm $= 1$.

$$= M_q(g) \|f\|_p = M_q(g) \text{ as claimed.}$$

This shows that

$$M_q(g) = \sup\left\{ \left|\int fg\right| \;\middle|\; \begin{array}{l} f \text{ bounded, measurable, vanishes} \\ \text{outside a set of finite} \\ \text{measure and } \|f\|_p = 1 \end{array} \right\}$$

Let $A = \{g \neq 0\}$. Then $A = \bigcup_{n=1}^{\infty} A_n$ where

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$$

and $m(A_n) < \infty$ $\forall n$. Why? take $A_n := A \cap [-n, n]$.

Now let $\phi_n$ be a sequence of ISF so that $\phi_n \to g$ pointwise and $|\phi_n| \le |g|$. Let $g_n := \phi_n \mathbb{1}_{A_n}$.

Then $g_n \to g$ pointwise, $|g_n| \le |g|$ and $g_n$ vanishes outside $A_n$.

Now let

$$f_n := \frac{|g_n|^{q-1}\, \mathrm{sgn}(g)}{\|g_n\|_q^{q-1}}$$

note that $f_n$ is bounded because $|g_n|$ is, that $f_n$ vanishes outside a set of finite measure, and since $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow q = p(q-1) \Rightarrow \|f_n\|_p = 1$.

Thus $f_n$ is a valid "test function" to integrate against $g$. (Note that $f_n$ may not be an ISF because $g$ could change sign infinitely many times. This is why the proof started by expanding the collection of things that $g$ could be integrated against.)

Fatou

$$\sqrt[q]{\int |g|^q} = \sqrt[q]{\int \underline{\lim}_{n\to\infty} |g_n|^q} \le \sqrt[q]{\underline{\lim}_{n\to\infty} \int |g_n|^q}$$

$$= \underline{\lim}_{n\to\infty} \left( \int |g_n|^{(q-1)} |g_n| \right)^{1/q}$$

since $|g_n| \le |g|$

def. of $f_n$ and $\frac{1}{p}+\frac{1}{q}=1$

$$= \underline{\lim}_{n\to\infty} \left( \int |f_n g_n| \right) \le \underline{\lim}_{n\to\infty} \int |f_n g|$$

$$= \underline{\lim}_{n\to\infty} \int f_n g \le M_q(g) < \infty$$

since $f_n g \geq 0$ because of the $\mathrm{sgn}(g)$ in $f_n$

This proves that $\|g\|_q \le M_q(g)$ and so $g \in L^q(\mathbb{R})$ as claimed. Now, by Hölder's inequality

$$\left|\int fg\right| \le \int |f||g| \le \|f\|_p \|g\|_q .$$

Hence $M_q(g) \le \|g\|_q$. This proves $M_q(g) = \|g\|_q$ as claimed //