Mat 357, March 4, 2005

<u>fact:</u> $\|f_n - f\|_1 \to 0$ does not imply that $f_n \to f$ pointwise almost everywhere

<u>ex:</u> $f_1 = 1_{[0,1]}$

$f_2 = 1_{[0,1/2]}$ $\quad f_3 = 1_{[1/2,1]}$

$f_4 = 1_{[0,1/4]}$, $f_5 = 1_{[1/4,1/2]}$, $f_6 = 1_{[1/2,3/4]}$, $f_7 = 1_{[3/4,1]}$

$f_8 = 1_{[0,1/8]}$, ...

Then $\|f_n\|_1 \to 0$ but $f_n \not\to 0$ pointwise almost everywhere.

Given $x \in [0,1]$, x will be in infinitely many diadic intervals and so there will be infinitely many n so that $f_n(x) = 1$. Hence $f_n(x) \not\to 0$.

<u>Note:</u> You can certainly find a <u>subsequence</u> so that $f_{n_k} \to 0$ almost everywhere. For example, $\{f_{2^n}\}_{n=0}^{\infty}$.

<u>thm:</u> Assume $\{f_n\} \subseteq L^1$ and $f \in L^1$ and $\|f_n - f\|_1 \to 0$. Then $\exists$ a subsequence f_{n_k} so that $f_{n_k} \to f$ pointwise almost everywhere.

<u>proof:</u> It suffices to show that if $\|f_n\|_1 \to 0$ then $\exists$ a subsequence $\{f_{n_k}\}$ so that $f_{n_k} \to 0$ a.e

We first show that if $\varepsilon > 0$ is fixed then
$$\lim_{n \to \infty} m(\{|f_n| \geq \varepsilon\}) = 0.$$

Let $E_{n,\varepsilon} := \{|f_n| \geq \varepsilon\}$. Then
$$0 \leq \varepsilon\, m(E_{n,\varepsilon}) \leq \int_{E_{n,\varepsilon}} |f_n| \leq \int |f_n| < \infty$$
and so $m(E_{n,\varepsilon}) < \frac{1}{\varepsilon}\|f_n\|_1$. Since $\|f_n\|_1 \to 0$ this shows $m(E_{n,\varepsilon}) \to 0$.

Given k, $\exists$ $N(k)$ so that for all $n \geq N(k)$
$$m\left(\left\{|f_n| \geq \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}$$
(Why? See previous paragraph w/ $\varepsilon = 1/2^k$)

Choose $N(k)$ so that
$$N(1) < N(2) < N(3) < \ldots$$

Let $g_k := f_{N(k)}$ and $E_k := \{ |g_k| \geq \frac{1}{2^k} \}$

$$F_k := \bigcup_{j \geq k}^{\infty} E_j \qquad F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$$

I claim that if $x \in \bigcup_{k=1}^{\infty} F_k^c$ then $g_n(x) \to 0$ as $n \to \infty$

since $x \in \bigcup_{k=1}^{\infty} F_k^c$ then $x \in F_{k_0}^c$ some k_0.

So $x \in \left(\bigcup_{j=k_0}^{\infty} E_j \right)^c = \bigcap_{j=k_0}^{\infty} E_j^c = \bigcap_{j=k_0}^{\infty} \{ |g_j| < \frac{1}{2^j} \}$

$\Rightarrow |g_j(x)| < \frac{1}{2^j}$ for all $j \geq k_0$

This shows that $g_n(x) \to 0$ as $n \to \infty$

Hence, if $D = \{ x \mid g_n(x) \not\to 0 \}$ then

$D \subseteq \left[\bigcup_{k=1}^{\infty} F_k^c \right]^c$. We finish the proof by

showing that $m\left(\left[\bigcup_{k=1}^{\infty} F_k^c \right]^c \right) = 0$.

$$\left[\bigcup_{k=1}^{\infty} F_k^c\right]^c = \bigcap_{k=1}^{\infty} [F_k^c]^c = \bigcap_{k=1}^{\infty} F_k$$

so by the continuity of measure,

$$m\left(\left[\bigcup_{k=1}^{\infty} F_k^c\right]^c\right) = m\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{k\to\infty} m(F_k)$$

$$= \lim_{k\to\infty} m\left(\bigcup_{j=k}^{\infty} E_j\right)$$

$$\leq \lim_{k\to\infty} \sum_{j=k}^{\infty} m(E_j)$$

$$< \lim_{k\to\infty} \sum_{j=k}^{\infty} \frac{1}{2^j} = \lim_{k\to\infty} \frac{1}{2^{k-1}} = 0,$$

as desired //

§12D The space $L^2(\mathbb{R})$.

Recall that $L^2(\mathbb{R}) := \{ f \mid f \text{ real-valued \& measurable}, \int |f|^2 < \infty \}$

Further $L^1(\mathbb{R}) \not\subseteq L^2(\mathbb{R})$ and $L^2(\mathbb{R}) \not\subseteq L^1(\mathbb{R})$

We will study $L^2(\mathbb{R})$ via its inner product where $(f,g) := \int fg$

This inner product induces the norm $\|f\| := \sqrt{(f,f)}$

Before considering $L^2(\mathbb{R})$ in depth, let's look at some general properties of vector spaces that have inner products.

<u>defn</u>: Let X be a vector space over $\mathbb{R}$. Then $(\cdot,\cdot) : X \times X \to \mathbb{R}$ is an inner product if the following hold:

$$(f,f) \geq 0 \quad \forall f \in X$$

$$(f,f) = 0 \iff f = 0$$

$$(f,g) = (g,f) \quad \forall f,g \in X$$

$$(af,g) = a(f,g) \quad \forall f,g \in X \quad \forall a \in \mathbb{R}$$

$$(f+g,h) = (f,h) + (g,h) \quad \forall f,g,h \in X$$

<u>claim</u>: If X is a real vector space and $(\cdot,\cdot)$ is an inner product then $\|f\| := \sqrt{(f,f)}$ is a norm on X.

<u>proof</u>: To see that $\|\cdot\|$ is a norm, we need to verify three things:

nondegeneracy: $\|f\| \geq 0 \quad \forall f$, $\|f\| = 0 \Leftrightarrow f = 0$

homogeneity: $\|\alpha f\| = |\alpha| \, \|f\|$

triangle inequality: $\|f+g\| \leq \|f\| + \|g\|$.

nondegeneracy:

$$\|f\| = \sqrt{(f,f)} \geq 0 \quad \text{since } (f,f) \geq 0$$

$$\|f\| = \sqrt{(f,f)} = 0 \Leftrightarrow (f,f) = 0 \Leftrightarrow f = 0 \checkmark$$

homogeneity:

$$\|\alpha f\| = \sqrt{(\alpha f, \alpha f)} = \sqrt{\alpha (f, \alpha f)}$$

$$= \sqrt{\alpha (\alpha f, f)} = \sqrt{\alpha^2 (f,f)}$$

$$= |\alpha| \sqrt{(f,f)} = |\alpha| \, \|f\| \checkmark$$

used the symmetry of $(\cdot,\cdot)$ & the property of $(\cdot,\cdot)$ & scalar mult.

triangle inequality:

$$\|f+g\|^2 = (f+g, f+g) = (f,f) + 2(f,g) + (g,g)$$

$$= \|f\|^2 + 2(f,g) + \|g\|^2$$

By the Cauchy-Schwarz inequality,

$$|(f,g)| \le \|f\| \, \|g\|$$

hence

$$\|f+g\|^2 \le \|f\|^2 + 2\|f\| \, \|g\| + \|g\|^2$$

$$= (\|f\| + \|g\|)^2$$

proving $\|f+g\| \le \|f\| + \|g\|$. //

Great! So all we need to prove is the

<u>Cauchy-Schwarz Inequality</u>: Let X be a real vector space, $(\cdot,\cdot)$ an inner product on X, and $\|\cdot\|$ the norm induced by $(\cdot,\cdot)$. Then

$$|(f,g)| \le \|f\| \, \|g\| \quad \forall f,g \in X.$$

<u>proof 1</u>: Fix $f,g \in X$.

define $\phi : \mathbb{R} \to [0,\infty)$ by $\phi(\lambda) := (f+\lambda g, f+\lambda g)$

Using the definition of inner product,

$$\phi(\lambda) = (f,f) + 2\lambda(f,g) + \lambda^2(g,g)$$

The graph of ϕ is a parabola with

$$\phi_{min} = \phi(\lambda_{min}) = \phi\left(-\frac{(f,g)}{(g,g)}\right)$$

$$= (f,f) - 2\frac{(f,g)}{(g,g)}(f,g) + \frac{(f,g)^2}{(g,g)^2}(g,g)$$

$$= (f,f) - \frac{(f,g)^2}{(g,g)}$$

Note that I found λ_{min} via calculus: $\frac{d\phi}{d\lambda} = 2(f,g) + 2\lambda(g,g) = 0$ when $\lambda = -(f,g)/(g,g)$.

Since $\phi_{min} \geq 0$ we see that

$$\phi_{min} = (f,f) - \frac{(f,g)^2}{(g,g)} \geq 0$$

$$\Rightarrow (f,f)(g,g) \geq (f,g)^2$$

$$\Rightarrow (f,g)^2 \leq \|f\|^2 \|g\|^2$$

$$\Rightarrow |(f,g)| \leq \|f\| \, \|g\|, \text{ as desired.} \;//$$