<u>defn:</u> A <u>Banach Space</u> is a complete normed vector space.

<u>Note:</u> Banach spaces can be finite dimensional like $(\mathbb{R}^n, \|\cdot\|_2)$ or they can be infinite dimensional like $(C([a,b]), \|\cdot\|_\infty)$.

<u>theorem:</u> $L^1(\mathbb{R})$ is complete

<u>proof:</u> Let $\{f_n\}_1^\infty$ be a Cauchy sequence in $L^1(\mathbb{R})$.

We seek $f \in L^1(\mathbb{R})$ so that $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

Given k, $\exists$ $N(k)$ so that $\|f_n - f_m\|_1 < \frac{1}{2^k}$ for all $m, n \geq N(k)$. Choose $N(k)$ so that in addition,

$$N(1) < N(2) < N(3) < \ldots < N(k)$$

Let $g_k := f_{N(k)}$. By construction, $\|g_k - g_{k+1}\|_1 < \frac{1}{2^k}$.

If we can find $f \in L^1(\mathbb{R})$ so that $\|g_n - f\|_1 \to 0$ as $n \to \infty$ then this will imply $\|f_n - f\|_1 \to 0$, and the proof will be finished. [Why? Given $\varepsilon > 0$, choose k such that $\frac{1}{2^k} < \varepsilon$.

choose $N(k_0) \geq N(k)$ so that $\|g_{\bar{k}} - f\|_1 < \varepsilon$. for all $\bar{k} \geq k_0$

Assume $n \geq N(k_0)$ Then

$\|f_n - f\|_1 \leq \|f_n - f_{N(k_0)}\|_1 + \|g_{k_0} - f\|_1 < \frac{1}{2^k} + \varepsilon < 2\varepsilon$]

We now construct a nondecreasing sequence of nonnegative integrable functions. We will show that this sequence converges.

$$h_1 := |g_1|$$

$$h_{n+1} := |g_1| + \sum_{k=1}^{n} |g_{k+1} - g_k| \qquad n \geq 1$$

Certainly, $0 \leq h_1 \leq h_2 \leq \ldots$ and each h_n is measurable. Given $x \in \mathbb{R}$, $\lim_{n\to\infty} h_n(x) =: h(x)$ exists if one allows $h(x) = \infty$. In this way, a pointwise limit of $\{h_n\}$ is defined.

Recall the monotone convergence theorem, "Suppose that $\{h_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of nonnegative measurable functions and h is the pointwise limit of h_n. Then $\lim_{n\to\infty} \int h_n = \int h$."

We want to show that h is integrable We do this by showing that $\int h_n \leq C < \infty$ for all n, hence $\int h < \infty \Rightarrow h \in L^1$:

$$\int h_{n+1} = \int |g_1| + \sum_{k=1}^{n} |g_{k+1} - g_k| = \|g_1\|_1 + \sum_{k=1}^{n} \|g_{k+1} - g_k\|_1$$

$$< \|g_1\|_1 + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \|g_1\|_1 + 1 =: C < \infty$$

This shows that h, the pointwise limit of h_n is integrable and hence finite almost everywhere

This proves that the series

$$g_1 + \sum_{k=1}^{\infty} (g_{k+1} - g_k)$$

is absolutely convergent almost everywhere. Since abs. conv $\Rightarrow$ convergent, we conclude that the partial sums

$$S_n = g_1 + \sum_{k=1}^{n} (g_{k+1} - g_k) = g_{n+1}$$

are convergent almost everywhere. We now define f:

$$f(x) = \begin{cases} \lim_{n \to \infty} g_n(x) & \text{if limit exists} \\ 0 & \text{otherwise} \end{cases}$$

f is measurable and $g_n \to f$ pointwise almost everywhere.

We now show that $\|g_n - f\|_1 \to 0$ as $n \to \infty$.

Since $f = g_1 + \sum_{k=1}^{\infty} (g_{k+1} - g_k) = g_n + \sum_{k=n}^{\infty} (g_{k+1} - g_k)$

$$f - g_n = \sum_{k=n}^{\infty} (g_{k+1} - g_k) = \lim_{N \to \infty} \sum_{k=n}^{N} (g_{k+1} - g_k)$$

$$\Rightarrow \quad |f(x) - g_n(x)| = \lim_{N \to \infty} \left| \sum_{k=n}^{N} (g_{k+1}(x) - g_k(x)) \right|$$

If $\psi_N := \left| \sum_{k=n}^{N} (g_{k+1} - g_n) \right|$ then the sequence ψ_N converges pointwise almost everywhere to $|f - g_n|$. We want to conclude that $|f - g_n| \in L^1$ and that $\|f - g_n\|_1$ is small. To do this, we apply the Lebesgue Dominated Convergence theorem. Note that

$$|\psi_N(x)| = \left| \sum_{k=n}^{N} (g_{k+1}(x) - g_n(x)) \right| \le \sum_{k=n}^{N} |g_{k+1}(x) - g_k(x)|$$

$$\le \sum_{k=n}^{\infty} |g_{k+1}(x) - g_k(x)| =: \psi(x)$$

We know $\psi \in L^1(\mathbb{R})$ because of the absolute convergence done earlier and

$$\|\psi\|_1 \le 2^{1-n}$$

So LDC applies and

$$\|f - g_n\|_1 = \int \lim_{N \to \infty} \left| \sum_{k=n}^{N} (g_{k+1} - g_k) \right| = \lim_{N \to \infty} \int |\psi_n| \le 2^{1-n}$$

Now we're almost done. We've got:

f measurable, $g_n \to f$ pointwise a.e.

$\|f - g_n\|_1 \to 0$ as $n \to \infty$.

We finally need to argue that $f \in L^1$.

Fix n. We know $g_n \in L^1$. And we've shown that $\|f-g_n\|_1 < \infty$ hence $f - g_n \in L^1$. Since L^1 is a vector space, $g_n + (f - g_n) \in L^1$, proving $f \in L^1$.

This finishes the proof of completeness. //

<u>defn</u>: $f : \mathbb{R} \to \mathbb{R}$ is <u>compactly</u> <u>supported</u> if $\exists\, M \geq 0$ such that $|x| > M \Rightarrow f(x) = 0$.

$C_0(\mathbb{R}) := \{ f \in C(\mathbb{R})$ and f has compact support$\}$

Now to prove that $C_0(\mathbb{R})$ is dense in $L^1(\mathbb{R})$.

<u>lemma</u>: If C is an open subset of $\mathbb{R}$ and B is a closed & bounded nonempty subset of C then there exists a compactly supported continuous function g such that $0 \le g \le 1$ and

$$x \in B \Rightarrow g(x) = 1$$
$$x \in C^c \Rightarrow g(x) = 0.$$

<u>proof</u>: homework!

<u>theorem</u>: $\overline{(C_0(\mathbb{R}), \|\cdot\|_1)} = (L^1(\mathbb{R}), \|\cdot\|_1)$. That is, given $f \in L^1(\mathbb{R})$ and $\varepsilon > 0$ there exists

$$g \in C_0(\mathbb{R}) \text{ such that } \|f-g\|_1 < \varepsilon.$$

<u>proof</u>: Since the ISF are dense in $L^1(\mathbb{R})$, $\exists$ an ISF f_0 such that $\|f-f_0\|_1 < \varepsilon/2$. Since $\|f-g\|_1 \le \|f-f_0\|_1 + \|f_0-g\|_1$, it suffices to find $g \in C_0(\mathbb{R})$ so that $\|f_0-g\|_1 < \varepsilon/2$.

By definition, $f_0 = \sum_{k=1}^{N} a_k \mathbb{1}_{A_k}$ where $a_k \neq 0$ and $m(A_k) < \infty$. And so, if we can find $g_k \in C_0(\mathbb{R})$ so that $\|g_k - a_k \mathbb{1}_{A_k}\|_1 \le \frac{\varepsilon}{2N}$ then

$g := \sum_{k=1}^{N} g_k \in C_0(\mathbb{R})$ will satisfy $\|f_0 - g\|_1 < \varepsilon/2$.

Fix k.

By §10D, property IV, "There exists a closed set B_k and an open set C_k such that $B_k \subseteq A_k \subseteq C_k$ and $m(C_k - B_k) < \frac{\varepsilon}{4|N||a_k|}$. If $m(A_k) < \infty$ then B_k can be chosen to be a bounded set."

We want to apply the lemma to B_k & C_k. The lemma required that B_k be closed, bounded, and nonempty. We already know B_k is closed & bounded. If it happens to be empty then let $B_k = \{a\}$ for some $a \in A_k$. This won't affect the measure of $C_k - B_k$ and will make B_k nonempty.

By the lemma, $\exists\, \psi_k \in C_0(\mathbb{R})$ so that $0 \le \psi_k \le 1$, $\psi_k = 1$ on B_k and $\psi_k = 0$ on C_k^c.

Let $f_k := a_k \psi_k \in C_0(\mathbb{R})$. It now remains to show $\|f_k - a_k \mathbb{1}_{A_k}\|_1 < \frac{\varepsilon}{2N}$.

$$\|f_k - a_k 1_{A_k}\|_1 = \int |f_k - a_k 1_{A_k}| = \int |a_k \psi_k - a_k 1_{A_k}|$$

$$= |a_k| \int |\psi_k - 1_{A_k}|$$

$$= |a_k| \left[\int_{B_k} |\psi_k - 1_{A_k}| + \int_{C_k - B_k} |\psi_k - 1_{A_k}| + \int_{C_k^c} |\psi_k - 1_{A_k}| \right]$$

$$= |a_k| \int_{C_k - B_k} |\psi_k - 1_{A_k}|$$

$$\leq |a_k| \int_{C_k - B_k} |\psi_k| + |1_{A_k}|$$

$$\leq |a_k| \, 2 \, m(C_k - B_k) < |a_k| \, 2 \, \frac{\varepsilon}{4N|a_k|}$$

$$= \frac{\varepsilon}{2N}$$ as desired. //