Mat 357 March 14, 2005

We now prove the covering lemma.

<u>Proof</u>: $C_1 := \sup\{|I| \mid I \in \mathcal{I}\} < \infty$.

where $\mathcal{I}$ is the family of open intervals whose union equals E.

Choose $I_1 \in \mathcal{I}$ such that $|I_1| \geq \frac{C_1}{2}$.

Now let $\mathcal{I}_2 := \{I \in \mathcal{I} \mid I \cap I_1 = \phi\}$.

let $C_2 := \sup\{|I| \mid I \in \mathcal{I}_2\}$

and if $\mathcal{I}_2 \neq \phi$ choose $I_2 \in \mathcal{I}_2$ so that $|I_2| \geq \frac{C_2}{2}$

Continue in this way where

$$\mathcal{I}_{n+1} := \{I \in \mathcal{I} \mid I \cap I_k = \phi \text{ for } k = 1..n\}$$

$$C_{n+1} := \sup\{|I| \mid I \in \mathcal{I}_{n+1}\}.$$

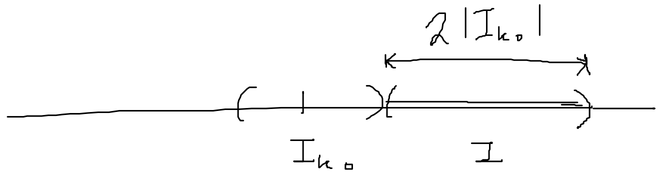
If $\mathcal{I}_{n+1}$ is empty stop. Otherwise, choose $I_{n+1} \in \mathcal{I}_{n+1}$ such that $|I_{n+1}| \geq \frac{C_{n+1}}{2}$.

case 1: for some n, $\mathcal{J}_{n+1} = \phi$.

If $I \in \mathcal{J}$ then this means that $I \cap I_k \neq \phi$ for some $k = 1 \dots n$. Let k_0 be the first such k. Hence $I \in \mathcal{J}_1, \mathcal{J}_2 \dots \mathcal{J}_{k_0}$ but $I \notin \mathcal{J}_{k_0+1}$.

Hence $|I| \leq c_1, c_2, \dots c_{k_0}$. $\Rightarrow |I_{k_0}| \geq \frac{c_{k_0}}{2} \geq \frac{|I|}{2}$.

Therefore $I \cap I_{k_0} \neq \phi$ and $|I| \leq 2|I_{k_0}|$



extreme case geometrically.

$\Rightarrow I \subseteq I_{k_0}^*$ where $I_{k_0}^*$ has the same midpoint as I_{k_0} and $|I_{k_0}^*| = 5|I_{k_0}|$.

Since $E = \bigcup I \Rightarrow E \subseteq \bigcup_{k=1}^{n} I_k^*$

$$\Rightarrow m(E) \leq \sum_1^n |I_k^*| = 5 \sum_1^n |I_k|$$

as desired.

<u>Case 2:</u> $\mathcal{J}_{n+1} \neq \emptyset$ for all n.

If $\sum_{n=1}^{\infty} |I_n| = \infty$ then the theorem is vacuously true.

Assume $\sum_{n=1}^{\infty} |I_n| < \infty$. Then $|I_n| \to 0$ as $n \to \infty$.

If $I \in \mathcal{J}$ then $\exists N$ so that $|I_n| < |I|/2$ for all $n \geq N$.

and since $|I_n| \geq \frac{a_n}{2}$ we have $\frac{a_n}{2} \leq \frac{|I|}{2}$ for all $n \geq N$.

$\Rightarrow I \notin \mathcal{J}_n$ $\forall n \geq N$.

$\Rightarrow I \cap I_k \neq \emptyset$ for some $k < N$.

Let k_0 be the first such k.

By the previous argument, $I \subseteq I_{k_0}^*$

And now,

$$E = \bigcup I \subseteq \bigcup_{k=1}^{\infty} I_k^*$$

$$\Rightarrow m(E) \leq m\left(\bigcup_{k=1}^{\infty} I_k^*\right) \leq \sum_{k=1}^{\infty} |I_k^*|$$

$$= 5 \sum_{k=1}^{\infty} |I_k|$$

as desired. //

Chapter 13 : Fourier Series

From a previous lifetime, we cavalierly write a 2π-periodic function f as

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}$$

where $\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$

In general, if f is L-periodic, we write

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ik \frac{2\pi}{L} x}$$

where $\hat{f}(k) = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-ik \frac{2\pi}{L} x} dx$

When are such expressions valid?

Notations; Conventions

- We will only consider 2π-periodic functions (since one can rescale space and view an l-periodic function as 2π-periodic.)
- The interval I is $[-\pi, \pi)$
- The norms $\|\cdot\|$ and $\|\cdot\|_1$ are defined as

$$\|f\| := \sqrt{\frac{1}{2\pi}\int_I |f|^2}$$

$$\|f\|_1 := \frac{1}{2\pi}\int_I |f|$$

- The basis vectors are

$$e_n := e^{+ikx}$$

Although I will frequently write

$$\int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, dx \quad \text{instead of} \quad \int_I f\,\overline{e_n}$$

looks like a Riemann integral but is really a Lebesgue integral. (abuse of notation.)

- $L^2(I) := \{ f$ that are measurable & complex-valued and $\int_I |f|^2 < \infty \}$

$L^p(I)$ for $1 \le p < \infty$ is defined analogously

- If f and g are in $L^2(I)$ then
$$(f, g) := \frac{1}{2\pi} \int_I f\bar{g}$$

defn: f is an integrable periodic function if $f \in L^1(I)$ and $f(x+2\pi) = f(x)$ for almost all $x \in \mathbb{R}$.

Q: Under what conditions are the coefficients $\hat{f}(k)$ finite?

A: If $f \in L^p(I)$ where $1 \le p \le 2$ then $\hat{f}(k)$ is finite

Case 1: $p = 2$

$$|\hat{f}(k)| = \left| \frac{1}{2\pi} \int_I f \bar{e}_k \right|$$
$$\le \frac{1}{2\pi} \int_I |f| |\bar{e}_k|$$
$$\le \frac{1}{2\pi} \sqrt{\int_I |f|^2} \sqrt{\int_I |\bar{e}_k|^2} \qquad \text{(Cauchy-Schwarz)}$$
$$= \frac{1}{2\pi} \sqrt{\int_I |f|^2} \sqrt{2\pi} = \|f\| < \infty$$

case 2: $1 < p < 2$

Hölder

$$|\hat{f}(k)| \le \frac{1}{2\pi}\int_I |f||\bar{e}_k| \le \frac{1}{2\pi}\sqrt[p]{\int_I |f|^p}\sqrt[q]{\int_I |e_k|^q}$$

$$= (2\pi)^{1/q - 1}\sqrt[p]{\int_I |f|^p}$$

$$= (2\pi)^{1/p}\sqrt[p]{\int_I |f|^p} < \infty$$

case 3: $p = 1$

$$|\hat{f}(k)| \le \frac{1}{2\pi}\int_I |f||\bar{e}_k| = \frac{1}{2\pi}\int_I |f| = \|f\|_1 < \infty$$