Mat 357 March 11, 2005

continuation of the proof of the fundamental theorem of calculus for $f \in L^1(\mathbb{R})$...

$$\text{let } A := \left\{ x \;\middle|\; \overline{\lim_{y \to x}} \left| \frac{F(y)-F(x)}{y-x} - f(x) \right| > 0 \right\}$$

If we prove that $m(A) = 0$ then we've proven that $F'(x)$ exists and equals $f(x)$ for almost all x.

$$\text{Let } A_\delta := \left\{ x \;\middle|\; \overline{\lim_{y \to x}} \left| \frac{F(y)-F(x)}{y-x} - f(x) \right| > \delta \right\}.$$

If $\delta_n = 1/n$ then $A_{\delta_1} \supseteq A_{\delta_2} \supseteq A_{\delta_3} \supseteq \ldots$

and $A = \bigcup_{n=1}^{\infty} A_{\delta_n}$. $\Rightarrow m(A) = m\left(\bigcup_{n=1}^{\infty} A_{\delta_n}\right) = \lim_{n \to \infty} m(A_{\delta_n})$.

We will prove that given $\varepsilon > 0$ $\exists M$ (indep of ε & δ) so that $m(A_\delta) \le M\varepsilon$ for all δ. It would then follow that $m(A) \le M\varepsilon$. Since ε is arbitrary, this inequality holds for all $\varepsilon \Rightarrow m(A) = 0$ as desired.

Fix $\varepsilon > 0$ and $\delta > 0$.

②

Choose $g \in C_c(\mathbb{R})$ so that $\|f-g\|_1 < \varepsilon\delta$

Let $G(x) := \int_a^x g(y)\,dy$. Given $y \neq x$ let I be the closed interval with endpoints x & y.

if $x<y$ then $I=[x,y]$ and $\frac{F(y)-F(x)}{y-x} = \frac{1}{|I|}\int_{[x,y]} |f| = \frac{1}{|I|}\int_I f$

then

$$\left| \frac{F(y)-F(x)}{y-x} - f(x) \right| \leq \frac{1}{|I|}\left| \int_I f-g \right| + \left| \frac{G(y)-G(x)}{y-x} - g(x) \right| + |g(x)-f(x)|$$

$$\leq \frac{1}{|I|}\int_I |f-g| + \left| \frac{G(y)-G(x)}{y-x} - g(x) \right| + |g(x)-f(x)|$$

$h := f-g \in L^1(\mathbb{R})$

The Hardy-Littlewood function $h^*(x)$ is defined via open intervals. Introducing $I_n = (x-1/n, y+1/n)$ we can use the Lebesgue Dominated convergence theorem to show that $\lim_{n\to\infty} \frac{1}{|I_n|}\int_{I_n} |h| = \frac{1}{|I|}\int_I |h|$. Since $\frac{1}{|I_n|}\int_{I_n} |h| \leq h^*(x)$, it follows

$$\Rightarrow \frac{1}{|I|}\int_I |f-g| \leq h^*(x)$$

If $x>y$, the same argument with $I=[y,x]$ leads to $\left| \frac{F(x-y)}{x-y} - f(x) \right| \leq h^*(x)$.

Hence

$$\left| \frac{F(y)-F(x)}{y-x} - f(x) \right| \le h^*(x) + \left| \frac{G(y)-G(x)}{y-x} - g(x) \right| + |h(x)|$$

$\Rightarrow$

$$\varlimsup_{y\to x} \left| \frac{F(y)-F(x)}{y-x} - f(x) \right| \le h^*(x) + \varlimsup_{y\to x} \left| \frac{G(y)-G(x)}{y-x} - g(x) \right| + |h(x)|$$

$$= h^*(x) + |h(x)|$$

Above, we used that the $\overline{\lim}$ of a sum is less than or equal to the sum of the $\overline{\lim}$. We also used that $h^*(x)$ and $|h(x)|$ are indep of y. And that G' exists and equals g.

If $a \le b + c$ and $b \le \delta/2$ & $c \le \delta/2$ then $a \le \delta$

Hence if $a > \delta$ then either $b > \delta/2$ or $c > \delta/2$.

Hence $A_\delta \subseteq \{h^* > \delta/2\} \cup \{|h| > \delta/2\}$

$\Rightarrow m(A_\delta) \le m(\{h^* > \delta/2\}) + m(\{|h| > \delta/2\})$

by the Hardy-Littlewood inequality, $m(\{h^* > \delta/2\}) \le \frac{5}{\delta/2} \|h\|_1 = \frac{10}{\delta} \|h\|_1$

Also, $0 \le \frac{\delta}{2} m(\{|h| > \frac{\delta}{2}\}) \le \int_{|h| > \delta/2} |h| \le \int |h| = \|h\|_1$

$\Rightarrow m(\{|h| > \frac{\delta}{2}\}) \le \frac{2}{\delta} \|h\|_1$

Hence $m(A_\delta) \leq \frac{10}{\delta}\|h\|_1 + \frac{2}{\delta}\|h_*\|_1 < \frac{12}{\delta}(\varepsilon\delta) = 12\varepsilon$

since $\|h\|_1 = \|f-g\|_1 < \varepsilon\delta$ by our choice of g.

We've now shown that $m(A_\delta) \leq 12\varepsilon$ independent of δ, as desired. //

We now prove the Hardy-Littlewood inequality using the following covering lemma from geometric measure theory.

<u>lemma:</u> Suppose $E \subseteq \mathbb{R}$ is measurable and E is the union of a family of open intervals of length $\leq c < \infty$. Then there exists a finite or countable subcollection of pairwise disjoint intervals $I_1, I_2, \ldots$ in this family such that

$$m(E) \leq 5 \sum |I_i|$$

<u>proof of Hardy-Littlewood Inequality:</u>

$E_\lambda := \{h^* > \lambda\}$.

Let $x \in E_\lambda$. Then $\exists$ open interval I_x containing x such that $\frac{1}{|I_x|}\int_{I_x} |h| > \lambda \Rightarrow |I_x| < \frac{1}{\lambda}\|h\|_1 =: C < \infty$.

Note that if $y \in I_x$ then $h^*(y) > \lambda$ too.

Hence $I_x \subseteq E_\lambda$. We now take the union over all x in E_λ. Hence

$$\bigcup_x I_x \subseteq E_\lambda \subseteq \bigcup_x I_x \Rightarrow E_\lambda = \bigcup_x I_x.$$

We have now proven that E_λ is the union of a family of open intervals each of whose length is $\leq$ a finite constant C.

By the covering lemma, there's a finite or countable subcollection that also covers E_λ.

Such that $m(E_\lambda) \leq 5 \sum |I_{x_i}|$

$$\leq \frac{5}{\lambda} \sum \int_{I_{x_i}} |h| \overset{\circledast}{=} \frac{5}{\lambda}\int_{\cup I_{x_i}} |h| \leq \frac{5}{\lambda}\|h\|_1.$$

//

$\circledast$ used the mutual disjointness of the subcollection.