Mat 357 Feb 28, 2005

We define a vector space over a field F as follows :

$$(X, F, +, \cdot)$$

the vectors, the field, vector addition, scalar multiplication

$(X, F, +, \cdot)$ is a vector space if

1) Any two vectors x & y in X uniquely determine a third vector $x+y$ in X and this vector addition satisfies

   a) $x+y = y+x \quad \forall x, y \in X$

   b) $(x+y)+z = x+(y+z) \quad \forall x, y, z \in X$

   c) $\exists\, 0 \in X$ such that $x+0 = x \quad \forall x \in X$

   d) given $x \in X$ $\exists\, y \in X$ such that $x+y = 0$. (call y "$-x$")

2) given $\alpha \in F$ and $x \in X$, α and x uniquely determine an element $\alpha x \in X$ and this scalar multiplication satisfies

   a) $\alpha(\beta x) = (\alpha\beta)x \quad \forall \alpha, \beta \in F, \forall x \in X$

   b) $1x = x \quad \forall x \in X$

3) $(\alpha+\beta)x = \alpha x + \beta x \quad \forall \alpha, \beta \in F \ \forall x \in X$

   $\alpha(x+y) = \alpha x + \alpha y \quad \forall \alpha \in F \ \forall x, y \in X$

Here are examples of real vectorspaces.
I'll let you deduce the "natural" vector addition and scalar multiplication.

$X = \mathbb{R}^n$

$X = \{ g \mid g \text{ is a real-valued ISF} \}$

$X = C([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is continuous} \}$

$X = C^k([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid \frac{d^j f}{dx^j} \text{ exists and is continuous } \forall\, 0 \le j \le k \}$

$X = C^\infty([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid \frac{d^j f}{dx^j} \text{ exists and is cont. for all } j \}$

$X = C(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}$

$C^k(\mathbb{R}), C^\infty(\mathbb{R})$ defined analogously

$X = C_0(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is cont, has compact support} \}$

$C_0^k(\mathbb{R}), C_0^\infty(\mathbb{R})$ defined analogously

$X = L^1(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ integrable} \}$

for $1 \le p < \infty$ also define

$$L^p(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ measurable and } \int |f|^p < \infty \}$$

Of the above, only $\mathbb{R}^n$ is a finite dimensional vector space. Some of the vector spaces are contained in each other:

$$C^\infty(\mathbb{R}) \subseteq C^k(\mathbb{R}) \subseteq C(\mathbb{R}) \subsetneq ISF$$

$$C_0(\mathbb{R}) \subseteq C(\mathbb{R})$$

what about $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$ if $p \neq q$? (think!)

Given a vector space, one can assign it a norm:

<u>defn:</u> $\|\cdot\| : X \to [0, \infty)$ is a norm if

1) $\|x\| \geq 0 \quad \forall x \in X$
2) $\|x\| = 0 \iff x = 0$
3) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$
4) $\|\alpha x\| = |\alpha| \, \|x\| \quad \forall \alpha \in F, \forall x \in X$

Once you have a norm, you can use the norm to generate a topology for the vector space

<u>defn:</u> $(X, \|\cdot\|)$ has a natural topology T where T is generated by the set of open sets $\{U = B_r(x) \mid 0 < r < \infty,\ x \in X\}$

where $B_r(x) := \{y \mid \|x-y\| < r\}$.

<u>Note:</u> if you have a norm on X you can use the norm to give X a metric:

$$d(x,y) := \|x-y\|.$$

<u>Fact:</u> If $(X, \|\cdot\|)$ is a normed vector space then the topology induced by the norm is the same as the metric space topology.

<u>Fact:</u> All normed vector spaces are metric spaces.

Not all vector spaces with metrics are normed vector spaces.

For example, $(\mathbb{R}^n, d)$ is a vector space w/ metric $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$. But it cannot

be normed since if $d(x,y) = \|x-y\|$ for some norm then that norm cannot be homogeneous (Specifically, $\|\alpha x\| = |\alpha| \, \|x\|$ will not hold if $\alpha = 2$.)

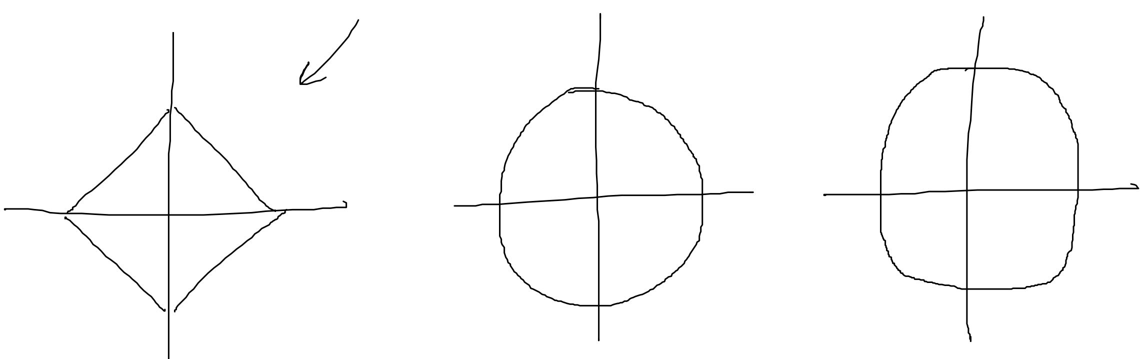
Examples of normed vector spaces:

$(\mathbb{R}^n, \|\cdot\|_p)$ where $1 \leq p < \infty$ and

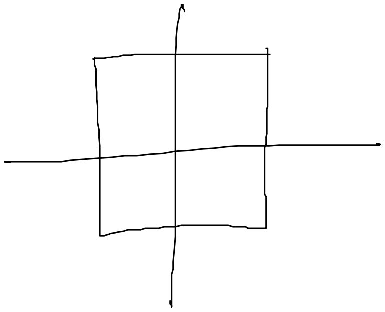
$$\|x\|_p := \sqrt[p]{|x_1|^p + |x_2|^p + \dots + |x_n|^p}$$

$(\mathbb{R}^n, \|\cdot\|_\infty)$ where $\|x\|_\infty := \max\{|x_1|, |x_2|, \dots |x_n|\}$

unit ball wrt $\|\cdot\|_1$



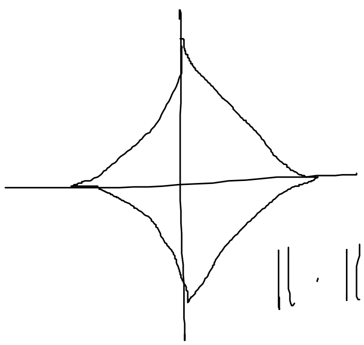
$\|\cdot\|_1$ unit ball $\quad$ $\|\cdot\|_2$ unit ball $\quad$ $\|\cdot\|_4$ unit ball



$\|\cdot\|_\infty$ unit ball.

as $p \to \infty$ the $\|\cdot\|_p$ unit ball $\to$ $\|\cdot\|_\infty$ unit ball

<u>fact:</u> for $p<1$, $\|\cdot\|_p$ isn't a norm. (why?)



$\|\cdot\|_{1/2}$ unit ball

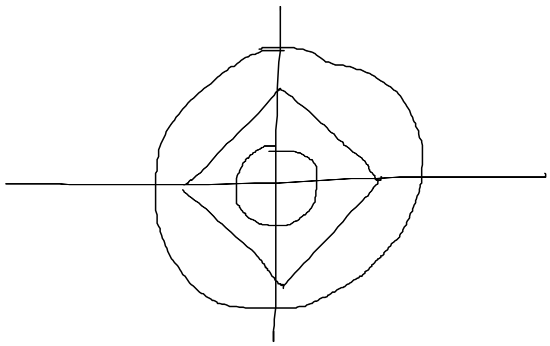
$\|\cdot\|$ & $|||\cdot|||$

<u>defn:</u> two norms on X are equivalent if $\exists\, c>0$ so that

$$c\|x\| \le |||x||| \le \frac{1}{c}\|x\| \quad \forall x \in X$$

i.e. you can take the $|||\cdot|||$ unit ball and put a small $\|\cdot\|$-ball inside it and put it inside a larger $\|\cdot\|$-ball.

e.g. $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on $\mathbb{R}^2$



fact: if $\|\cdot\| \sim |||\cdot|||$ then $(X, \|\cdot\|)$ and $(X, |||\cdot|||)$ will have precisely the same open sets.

fact: Let X be a finite dimensional vector space. Let $\|\cdot\|$ and $|||\cdot|||$ be two norms on X. Then $\|\cdot\| \sim |||\cdot|||$ (why? HW!)

conclusion: finite-dimensional normed vector spaces are boring.

What are some infinite dimensional normed vector spaces?

$(C([a,b]), \|\cdot\|_\infty)$

where $\|f\|_\infty = \max\limits_{x \in [a,b]} |f(x)|$

$(C([a,b]), \|\cdot\|_p)$

where $p \in [1,\infty)$ and

$\|f\|_p := \sqrt[p]{\int |f|^p}$

two things to be careful about:

1) How do you prove these norms satisfy the triangle inequality?
2) How do you prove that

$\|f\|_p = 0 \Leftrightarrow f = 0$?

Q: are $\|\cdot\|_p$ and $\|\cdot\|_q$ equivalent norms on $C([a,b])$? (think!)

Consider the two norms on $C^1([a,b])$ where

$\|f\|_{2,1} := \sqrt{\int |f|^2 + |f'|^2}$

$\|f\|_{2,0} := \sqrt{\int |f|^2}$

prove or disprove: $\|\cdot\|_{2,1} \sim \|\cdot\|_{2,0}$ on $C^1([a,b])$

Fact:

if $X = \{g \mid g: \mathbb{R} \to \mathbb{R}$ was ISF$\}$

then $(X, \|\cdot\|_p)$ $1 \le p < \infty$ is not a

normed vector space. Why? It's not

true that $\|f\|_p = 0 \Leftrightarrow f = 0$ for $f \in X$

if $X = L^1(\mathbb{R})$

then $(X, \|\cdot\|_1)$ is not a normed vector

space, for the same reason

Similarly, $(L^p(\mathbb{R}), \|\cdot\|_p)$ is not a normed

vector space

What's the fix? We will introduce an equivalence

relation on $X = \{g \mid g$ an ISF$\}$ as follows:

$f \sim g \Leftrightarrow f = g$ a.e. Then $(X/\sim, \|\cdot\|_p)$

is a normed vector space where

$\|[f]\|_p := \|f\|_p$ where f is a representative of the equivalence class $[f]$.

Similarly, $\left(L^1(\mathbb{R})/\sim, \|\cdot\|_1\right)$ and $\left(L^p(\mathbb{R})/\sim, \|\cdot\|_p\right)$

are normed vector spaces.

<u>fact:</u> I will use $L^1(\mathbb{R})$ as short-hand for $L^1(\mathbb{R})/\sim$

$L^p(\mathbb{R})$ as short hand for $L^p(\mathbb{R})/\sim$

We do not have any such quotienting for the $C([a,b])$, $C^k([a,b])$, $C^\infty([a,b])$, etc spaces. If we say continuous we mean continuous <u>everywhere</u>.

Okay... we now have normed vector spaces. We move on to ask whether they're <u>complete</u>.

<u>defn:</u> $(X, \|\cdot\|)$ is complete if every cauchy sequence converges and the limit is in X.

i.e if $\{x_n\}$ is Cauchy then $\exists\, x \in X \ni$

$$\|x_n - x\| \to 0.$$

<u>fact:</u> if $(X, \|\cdot\|)$ is complete and $(X, |||\cdot|||)$ is not complete then $\|\cdot\| \not\approx |||\cdot|||$ (i.e the norms cannot be equiv)

If $(X, \|\cdot\|)$ is not complete, it's very natural to ask what its completion is. If the completion of $(\mathbb{Q}, |\cdot|)$ is $\mathbb{R}$, what's the completion of $(C([a,b]), \|\cdot\|_p)$?

I'll use the notation $\overline{(C([a,b]), \|\cdot\|_p)}$ for the completion of $(C([a,b]), \|\cdot\|_p)$

things you can prove to yourself:

$X = \{ g \mid \text{ISF supported in } [a,b] \}$

then $\overline{(X, \|\cdot\|_1)} = (L^1([a,b]), \|\cdot\|_1)$

$Y = \{ g \mid \text{ISF} \}$

then $\overline{(Y, \|\ \|_1)} = (L^1(\mathbb{R}), \|\cdot\|_1)$

$(C([a,b]), \|\cdot\|_\infty)$ is complete. Why?
because convergence with respect to the $\|\cdot\|_\infty$ is the same as uniform convergence And the uniform limit of continuous functions is continuous.

$(C^1([a,b]), \|\cdot\|_\infty)$ is not complete. Why?
You can construct a sequence of C^1 functions that converge uniformly to the absolute value function (if $a<0<b$, for example) which is continuous but not C^1.

$(C([a,b]), \|\cdot\|_p)$ with $1 \le p < \infty$ is not complete. Why? You can construct a sequence of continuous functions so that $\|f_n - f\|_p \to 0$ but f is a step function. (Which isn't in $C([a,b])$ since it's not continuous.)

Things that require some work:

$$\overline{\left(C^k([a,b]), \|\cdot\|_\infty\right)} = \left(C([a,b]), \|\cdot\|_\infty\right) \quad \text{for } k \geq 1$$

$$\overline{\left(C^\infty([a,b]), \|\cdot\|_\infty\right)} = \left(C([a,b]), \|\cdot\|_\infty\right)$$

if $\|f\|_{\infty,k} := \max\limits_{x \in [a,b]} |f(x)| + |f^{(1)}(x)| + |f^{(2)}(x)| + \dots + |f^{(k)}(x)|$

where $f^{(j)} = \dfrac{\partial^j f}{\partial x^j}$;

then $\overline{\left(C^l([a,b]), \|\cdot\|_{\infty,k}\right)} = \left(C^k([a,b]), \|\cdot\|_{\infty,k}\right)$ for $k > l$

similarly for the completion of $C^\infty([a,b])$

What about L^p norms? for $1 \leq p < \infty$

$$\overline{\left(C([a,b]), \|\cdot\|_p\right)} = \left(L^p([a,b]), \|\cdot\|_p\right)$$

$$\overline{\left(C^k([a,b]), \|\cdot\|_p\right)} = \left(L^p([a,b]), \|\cdot\|_p\right)$$

$$\overline{\left(C^\infty([a,b]), \|\cdot\|_p\right)} = \left(L^p([a,b]), \|\cdot\|_p\right)$$

$$\overline{\left(C_0(\mathbb{R}), \|\cdot\|_p\right)} = \left(L^p(\mathbb{R}), \|\cdot\|_p\right)$$

$$\overline{\left(C_0^k(\mathbb{R}), \|\cdot\|_p\right)} = \left(L^p(\mathbb{R}), \|\cdot\|_p\right)$$

$$\overline{\left(C_0^\infty(\mathbb{R}), \|\cdot\|_p\right)} = \left(L^p(\mathbb{R}), \|\cdot\|_p\right)$$

These statements become far more interesting and useful if you read them as follows:

$$\overline{(X, \|\cdot\|)} = (Y, \|\cdot\|)$$

means "X is dense in Y with respect to the norm $\|\cdot\|$." This means that if you're trying to prove something about $(Y, \|\cdot\|)$ you may be well-advised to try and prove it for elements of X and then see if you can leverage off of this. (Elements of X may be much nicer.)

For example, to prove that $\|\cdot\|_p$ satisfies the triangle inequality for $L^p(\mathbb{R})$ you're well-advised to prove that if f and g are ISF then $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ and then use the fact that $\overline{(\text{ISF}, \|\cdot\|_p)} = (L^p, \|\cdot\|_p)$ to prove the triangle inequality for $f, g \in L^p(\mathbb{R})$.