MAT 357, Feb 25 2005

<u>defn:</u> a <u>null set</u> is a measurable set with measure zero.

<u>fact:</u> If $A \subseteq B$ and $B$ is a null set then $A$ is a null set.

<u>proof:</u> It suffices to show that for all subsets $E$ one has

$$m^*(A \cap E) + m^*(A^c \cap E) = m^*(E).$$

The $\geq$ direction is trivial, so it suffices to show that

$$m^*(A \cap E) + m^*(A^c \cap E) \leq m^*(E).$$

$B$ is null $\Rightarrow m(B) = m^*(B) = 0$

$A \subseteq B \quad \Rightarrow m^*(A) \leq m^*(B) = 0 \Rightarrow m^*(A) = 0$

$A \cap E \subseteq A \subseteq B \quad \Rightarrow m^*(A \cap E) = 0$ analogously

hence it suffices to show that

$$m^*(A^c \cap E) \leq m^*(E).$$

This is immediate since $A^c \cap E \subseteq E$. //

<u>defn:</u> $f = g$ almost everywhere if $m(\{f \neq g\}) = 0$

prop:

1) assume $f$ is measurable, real-valued and $g$ is real-valued with $f=g$ a.e. Then $g$ is a measurable function

2) Assume $f \geq 0$ is a measurable function. Then

$$\int f = 0 \iff f = 0 \text{ a.e.}$$

3) If $f$ is an integrable, real valued functions then $m(\{f = \pm\infty\}) = 0$.

proof:

1) ~~bu~~ We need to show that $\{g > a\} =: B$ is a measurable set for all $a \in \mathbb{R}$. By assumption, $A := \{f > a\}$ is measurable. Since $f=g$ a.e., $A$ and $B$ agree up to a set of measure zero i.e. $m(A \Delta B) = 0$. It follows that $B$ is measurable. (Why? this is one of your HW problems.)

2) ($\Rightarrow$) Assume $\int f = 0$. Let $A = \{f > 0\} = \bigcup_{n=1}^{\infty} \{f > 1/n\} =: \bigcup_{n=1}^{\infty} A_n$

by continuity, $m(A) = \lim_{n\to\infty} m(A_n)$.

We know $\quad 0 \leq \frac{1}{n} m(A_n) < \int_{A_n} f \leq \int f = 0$

Hence $m(A_n) = 0 \quad \forall n \quad \Rightarrow m(A) = 0$ as desired.

($\Leftarrow$) Assume $m(\{f>0\})=0$. By definition,

$$\int f = \sup\{\int g \mid 0 \le g \le f,\ g \text{ an ISF}\}.$$

Let $g$ be such an ISF. Then

$$g = \sum_{k=1}^{N} a_k \mathbb{1}_{A_k}$$

where $a_k > 0$ and $\{A_k\}$ are pairwise disjoint and measurable. If $m(A_{k_0})>0$ for some $k_0$ then $\int g \geq a_{k_0} m(A_{k_0}) > 0 \Rightarrow \int f > 0$ ※. Hence $m(A_k)=0$ for $k=1 \dots N$ and $\int g = 0$. This is true for all candidate ISF, hence $\int f = 0$.

3) First, assume $f \geq 0$.

$$B := \{f=\infty\} = \bigcap_{n=1}^{\infty} \{f>n\} =: \bigcap_{n=1}^{\infty} B_n$$

by continuity, $m(B) = \lim_{n\to\infty} m(B_n)$

Now $\quad n\, m(B_n) < \int_{B_n} f \leq \int f < \infty \quad$ since

$f$ is integrable. Hence $m(B_n) < \frac{1}{n}\int f$ and $\lim_{n\to\infty} m(B_n) = 0$, proving $m(\{f=\infty\}) = 0$.

Now, assume $f$ is real valued. Then $f = f^+ + f^-$ and $\int |f| = \int f^+ + \int f^- < \infty$ since $f$ is integrable.

Hence $f^+ \geq 0$, $f^+$ measurable, $\int f^+ < \infty$. By previous part, $m(\{f^+ = \infty\}) = m(\{f = \infty\}) = 0$. Similarly, $m(\{f^- = \infty\}) = m(\{f = -\infty\}) = 0$. Hence $m(\{f = \pm\infty\}) = 0$, as claimed //

We will use this lemma to prove that if a function $f$ is Riemann Integrable on an interval $[a,b]$ then it's Lebesgue Integrable on $[a,b] =: I$ and

$$\int_a^b f = \int_I f \quad \leftarrow \text{Lebesgue Integral}$$

Riemann Integral.

Note that while definite integrals agree, improper integrals don't necessarily. E.g

$$\int_0^\infty \frac{\sin(x)}{x}\, dx = \frac{\pi}{2} \qquad \text{but}$$

$$\int_0^\infty \left[\frac{\sin(x)}{x}\right]_+ dx = \infty \qquad \text{and so}$$

$$\int_{[0,\infty)} \frac{\sin(x)}{x} \quad \text{is not integrable.}$$

Lebesgue Integrals are blind to cancellation effects due to oscillation (see above) or symmetry. So this makes principal values tricky and the like. (These things show up in harmonic analysis, among other places.)

<u>defn</u>: A bounded, ~~real-valued~~ function $f:[a,b]\to\mathbb{R}$ is <u>Riemann Integrable</u> if $\exists$ two sequences of step functions $\{g_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ so that $g_n\le f\le h_n$ on $[a,b]$ for all $n$ and $\lim_{n\to\infty}\int h_n-g_n=0$. We then define $\int_a^b f := \lim_{n\to\infty}\int_I g_n = \lim_{n\to\infty}\int_I h_n$

<u>theorem</u>: If $f$ is Riemann integrable on $I=[a,b]$ then $f$ is integrable on $I$ and the Lebesgue integral $\int_I f$ equals the Riemann integral $\int_a^b f$.

<u>proof</u>: By assumption, $\exists$ step functions $\{g_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ on $I$ such that $g_n\le f\le h_n$ $\forall n$ and

$$\lim_{n\to\infty}\int_I g_n = \lim_{n\to\infty}\int_I h_n = \int_a^b f$$

We construct a nondecreasing sequence of ISF: $\tilde{g}_n := g_1 \vee g_2 \vee \dots \vee g_n$

and a nonincreasing sequence of ISF:

$$\tilde{h}_n := h_1 \wedge h_2 \wedge \dots h_n$$

By construction, $\tilde{g}_n \le f \le \tilde{h}_n$ and

$$0 \le \lim_{n\to\infty} \int_I \tilde{h}_n - \tilde{g}_n \le \lim_{n\to\infty} \int_I h_n - g_n = 0$$

Hence $\displaystyle\lim_{n\to\infty} \int_I \tilde{g}_n = \lim_{n\to\infty} \int_I \tilde{h}_n = \int_a^b f$

For each $x \in I$, the sequence $\{\tilde{g}_n(x)\}$ is nondecreasing and bounded above by $f(x)$. Hence $\lim_{n\to\infty} \tilde{g}_n(x)$ exists, call this limit $g(x)$. Similarly, construct $h$ the pointwise limit of $\tilde{h}_n$. $h$ and $g$ are measurable since they're limits of measurable function. Also, $|\tilde{g}_n| \le \phi := \tilde{h}_1 \vee (-\tilde{g}_1)$ and $\phi$ is an integrable function. Hence the Lebesgue Dominated Convergence theorem applies and

$$\lim_{n\to\infty} \int_I \tilde{g}_n = \int_I g \quad \text{and} \quad \lim_{n\to\infty} \int_I \tilde{h}_n = \int_I h.$$

We know that $\lim_{n\to\infty} \int_I \tilde{g}_n = \int_I \tilde{h}_n \lim_{n\to\infty}$ hence

$\int_I g = \int_I h \Rightarrow \int_I h - g = 0 \Rightarrow h = g$ almost everywhere

by the lemma. Since $g \leq f \leq h$, we find that

$h = f$ almost everywhere. Again by the lemma,

this implies that $f$ is measurable.

Now to show that $f$ is integrable.

<u>case 1</u>: $f \geq 0$

then $\int_I f := \sup\{ \int_I \bar{g} \mid \bar{g}$ an ISF, $0 \leq \bar{g} \leq f \}$.

Since each $\tilde{g}_n$ is a candidate, $\lim_{n\to\infty} \int \tilde{g}_n \leq \int_I f$.

Hence $\int_I h \leq \int_I f$. On the other hand, for

each candidate $\bar{g}$, one has $\bar{g} \leq f \leq h$. Hence

$\int_I f \leq \int_I h$. This proves $\int_I f = \int_I h = \lim_{n\to\infty} \int_I \tilde{h}_n = \int_a^b f$

as desired.

<u>case 2</u>: $f$ real-valued.

Then $f = f^+ - f^-$ where $f^+, f^- \geq 0$ are measurable. Since $f$ is Riemann integrable, so are $f^+$ and $f^-$ via $\tilde{g}_n^+ \leq f^+ \leq \tilde{h}_n^+$ and $\tilde{g}_n^- \leq f^- \leq \tilde{h}_n^-$. Further, $\int_a^b f^+ = \lim_{n\to\infty} \int_I \tilde{g}_n^+ = \int_I g^+ = \int_I h^+ = \int_I f^+$

analogously w/ $\int_a^b f^-$ (both by case 1)

Hence $\int_I f = \int_I f^+ - \int_I f^- = \int_a^b f^+ - \int_a^b f^- = \int_a^b f^+ - f^- = \int_a^b f$, as desired //

Super-cool theorem relating Riemann Integrability to measure theory!

<u>thm 12.3</u>: A bounded, measurable real-valued function on $[a,b]$ is Riemann integrable if and only if it's continuous almost everywhere in $[a,b]$.