We now prove our first limit theorem.

__Lemma 11.5:__ Suppose g is a nonnegative ISF and $\{f_n\}$ is a sequence of measurable functions such that

1) $g \geq f_1 \geq f_2 \geq f_3 \geq \dots \geq 0$
2) $f_n \to 0$ pointwise everywhere

Then $\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n = 0$.

__Proof__: Since g is an ISF, $g = \sum_{i=1}^{n_0} a_i \mathbb{1}_{A_i}$ where $a_i > 0$ and the measurable sets A_i are pairwise disjoint and have finite measure.

So $g \leq \max\{a_1, a_2, .. a_{n_0}\} = M$ and $m(\{g>0\}) < \infty$.

Since $0 \leq f_n \leq g$ $\forall n$, the sequence $\{f_n\}$ is uniformly bounded and is converging pointwise on a set of finite measure.

Given $\varepsilon > 0$, let $A_n := \{f_n > \varepsilon\}$, and $A := \{g > 0\}$. By construction,

$$A \supset A_1 \supset A_2 \supset A_3 \quad \text{and} \quad \bigcap_{i=1}^{\infty} A_i = \emptyset$$

(Why? Since $f_n \geq f_{n+1}$, if $f_{k+1}(x) > \varepsilon$ then $f_k(x) > \varepsilon$. So $A_{k+1} \subset A_k$. Also, if $x \in \bigcap_1^{\infty} A_i$ then $f_i(x) > \varepsilon$ $\forall i \Rightarrow \lim_{i \to \infty} f_i(x) \neq 0$. This contradicts the $f_n \to 0$ pointwise everywhere.)

By continuity (page 139, IV) we have

$$0 = m(\emptyset) = m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} m(A_n).$$

Choose N_ε so that $n \geq N_\varepsilon \Rightarrow m(A_n) < \varepsilon$.

Then $f_n \leq \varepsilon$ on $A - A_n$ for all $n \geq N_\varepsilon$

Now,

$$0 \leq \int f_n = \int_{A^c} f_n + \int_{A \cap A_n^c} f_n + \int_{A_n} f_n$$

$$= \int_{A \cap A_n^c} f_n + \int_{A_n} f_n \qquad \text{since } f_n \equiv 0 \text{ on } A^c$$

$$\leq \varepsilon\, m(A \cap A^c) + M\, m(A_n)$$

$$< \varepsilon\, m(A) + M\varepsilon \quad \text{if } n \geq N_\varepsilon.$$

This proves that $\int f_n \to 0$, as desired //

<u>note:</u> if $f_n \to 0$ almost everywhere then we have $m\left(\bigcap_{n=1}^{\infty} A_n\right) = 0$ since $x \in \bigcap_{n=1}^{\infty} A_n$ implies $x \notin C =$ "points at which $f_n \to 0$".

$\Rightarrow \bigcap_{n=1}^{\infty} A_n \subseteq A - C$ and we know $m(A-C) = 0$.

Since $m\left(\bigcap_1^{\infty} A_n\right) = 0$ we still have $m(A_n) < \varepsilon$ for $n \geq N_\varepsilon$, which is all we used. So the lemma would still be true.

Now we use this lemma to prove our big theorem:

<u>thm 11.6</u> (Lebesgue Dominated Convergence)
Suppose that $\{f_n\}_1^{\infty}$ is a sequence of measurable functions such that for all x, $\lim_{n\to\infty} f_n(x) =: f(x)$. Suppose also that all the f_n are dominated by an integrable function g in that $|f_n(x)| \leq g(x)$ $\forall x$.

Then $\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n = \int f$.

Proof:

Step 1 Assume $f = 0$ and $f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0$.

Given $\varepsilon > 0$, $\exists$ ISF g_1 such that $0 \leq g_1 \leq g$ and $\int g - \varepsilon < \int g_1 \Rightarrow 0 \leq \int g - g_1 < \varepsilon$. (Why? because g is integrable.) Note that $\{g_1 > 0\}$ has finite measure. We use g_1 to create a new sequence $\tilde{f}_n := f_n \wedge g_1$.

By construction, $g_1 \geq \tilde{f}_1 \geq \tilde{f}_2 \geq \cdots \geq 0$ and $\tilde{f}_n \to 0$ pointwise everywhere. By Lemma 11.5,

$$\lim_{n\to\infty} \int \tilde{f}_n = 0. \Rightarrow \exists N_\varepsilon \ni 0 \leq \int \tilde{f}_n < \varepsilon \text{ for } n \geq N_\varepsilon$$

Note that $f_n = f_n \wedge g_1 + (f_n - f_n \wedge g_1)$

$$\leq f_n \wedge g_1 + (g - g_1)$$

(Why? because $f_n \leq g$ and $f_n \wedge g_1 \leq g$)

Hence for $n \geq N_\varepsilon$,

$$0 \leq \int f_n \leq \int f_n \wedge g_1 + \int g - g_1 = \int \tilde{f}_n + \int g - g_1 < 2\varepsilon.$$

Step 2: Assume $f_n \geq 0$ and $f_n \to 0$

Define $g_n := \sup\{f_n, f_{n+1}, \dots\}$ then $g_1 \geq g_2 \geq g_3 \geq \cdots \geq 0$,

g_n is measurable $\forall n$, $g_n \to 0$ pointwise everywhere.

and $0 \leq g_n \leq g$. By Step 1,

$$\lim_{n\to\infty} \int g_n = 0.$$

Since $0 \leq \int f_n \leq \int g_n \to 0$ this implies

$$\lim_{n\to\infty} \int f_n = 0.$$

Step 3: The general case. Let $g_n := |f_n - f|$.

Then $g_n \geq 0$, g_n is measurable,

and $g_n \to 0$ pointwise everywhere. Also,

$$0 \leq g_n \leq |f_n| + |f| \leq 2g$$

so g_n is bounded by an integrable function.

By Step 2, $0 \leq \int g_n \to 0$ Hence

$$0 \leq \left| \int f_n - \int f \right| \leq \int |f_n - f| = \int g_n \to 0$$

proving that $\int f_n \to \int f$, as desired. //