Mat 357   Feb 21, 2005

Recall that

$\int f$ is shorthand for $\int f(x)\,dx$ where dx is from the Lebesgue measure on $\mathbb{R}$ (constructed in §10C) and $\int$ is the Lebesgue integral.

Recall that if $f \geq 0$ is a measurable, real-valued function then

$$\int f := \sup\left\{ \int g \;\middle|\; \begin{array}{l} g \text{ is an integrable simple} \\ \text{function with } 0 \leq g \leq f \end{array} \right\}$$

If f is measurable and real-valued then

$$\int f := \int f^+ - \int f^- \text{ where}$$

$$f^+ := f \vee 0 \text{ and } f^- := -(f \wedge 0)$$

Finally, recall that a measurable, real-valued function is <u>integrable</u> if

$$\int |f| < \infty.$$

note! $\int |f| < \infty \iff \int f^+ < \infty$ and $\int f^- < \infty$.

One of the main reasons the Lebesgue integral was introduced was because the Riemann integral does not behave well with respects to limits.

Consider $f = \mathbb{1}_{\mathbb{Q}\cap[0,1]}$ $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}\cap[0,1] \\ 0 & \text{otherwise} \end{cases}$

You can find a sequence of functions $\{f_n\}_1^\infty$ so that $f_n \to f$ and

$$\int_{RI} f_n(x)\,dx \to 0$$

Similarly, you can find a sequence of functions $\{g_n\}_1^\infty$ so that $g_n \to f$ and

$$\int_{RI} g_n(x)\,dx \to 1$$

So you can't take the limit under the Riemann integral. ☹

Recall, however, that if $f_n \to f$ <u>uniformly</u> then you can take the limit under the Riemann Integral

Our goal is to show that Lebesgue Integrals behave well with respect to limits. Specifically, we want to show something like

" $f_n \to f$ pointwise almost everywhere

& one extra condition

$$\Rightarrow \lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n = \int f."$$

To do this, we'll first start by studying Egorov's theorem

<u>Egorov's Theorem</u>: Let $\{f_n\}$ be a sequence of measurable functions on a measurable set E. Assume $f_n \to f$ pointwise almost everywhere on E and assume $m(E) < \infty$. Then given $\delta > 0$ $\exists$ a measurable set $E_\delta \subset E$ such that

1) $m(E_\delta) > m(E) - \delta$
2) $f_n \to f$ uniformly on E_δ.

Note: Egorov's theorem says that pointwise convergence of measurable functions means that except for a set of small measure ($E - E_\delta$) one really has uniform convergence. This will make taking the limit under the integral sign more natural.

note: for statement & proof of Egorov's theorem, see "Introductory Real Analysis" by Kolmogorov & Fomin, pp 290 - 291

Proof of Egorov's theorem:

First, the limiting function f is measurable by Beals' prop 11.2

Let $E_n^M := \bigcap_{i \geq n} \{ |f_i - f| < 1/M \}$. Thus for fixed M and n, E_n^M is the set of points $x \in E$ such that $|f_i(x) - f(x)| < 1/M$ for all $i \geq n$. Note that E_n^M is the countable intersection of measurable sets and is therefore a measurable set.

Now, let $E^M := \bigcup_{n=1}^{\infty} E_n^M$. By the comments at the end of this proof, if $G :=$ "the points in E at which one has pointwise convergence", then

$$G \subseteq E^M \subseteq E$$

but it's not true that G has to equal E^M.

By construction, $E_1^M \subseteq E_2^M \subseteq E_3^M \subseteq \dots E^M$ and so by Beals p 139, property IV,

$$m(E^M) = m\left(\bigcup_{n=1}^{\infty} E_n^M\right) = \lim_{n\to\infty} m(E_n^M).$$

Thus, given $\delta > 0$ $\exists\, n_0(M)$ such that

$$m\left(E^M - E^M_{n_0(M)}\right) < \delta/2^M$$

Now let $E_\delta := \bigcap_{M=1}^{\infty} E^M_{n_0(M)}$. By construction, E_δ is measurable and $m(E - E_\delta) = m\left(E - \bigcap_{M=1}^{\infty} E^M_{n_0(M)}\right)$

$$= m\left(\bigcup_{M=1}^{\infty} \left(E - E^M_{n_0(M)}\right)\right)$$

$$\leq \sum_{M=1}^{\infty} m\left(E - E^M_{n_0(M)}\right)$$

$$= \sum_{M=1}^{\infty} m\left(E^M - E^M_{n_0(M)}\right) \quad \circledast$$

$$< \sum_{M=1}^{\infty} \delta/2^M = \delta.$$

This shows $m(E - E_\delta) < \delta \Rightarrow m(E_\delta) > m(E) - \delta$.

In $\circledast$ I used $m(E - A) = m(E^M - A)$. This is true because $C \subseteq E^M \subseteq E$ and we were told that $m(E - C) = 0$. $\Rightarrow m(E - E^M) = 0$

$$\Rightarrow m(E - A) = m(E^M - A)$$

It remains to prove that f_n converges uniformly to f on E_δ.

$$x \in E_\delta \Rightarrow x \in E^M_{n_0(M)} \text{ for all } M$$

$$\Rightarrow |f_i(x) - f(x)| < 1/M \text{ for all } M \text{ and all } i > n_0(M).$$

Let $x \in E_\delta$ and $\varepsilon > 0$. Choose $\tilde{M}$ such that $1/\tilde{M} < \varepsilon$. Since $x \in E_\delta$, we know $x \in E^{\tilde{M}}_{n_0(\tilde{M})}$. Thus $|f_i(x) - f(x)| < \frac{1}{\tilde{M}} < \varepsilon$ for all $i > n_0(\tilde{M})$. This will hold for <u>any</u> $x \in E_\delta$, proving uniform convergence on E_δ, as desired. //

It remains to prove my claims concerning

C = "the points in E at which f_n converges pointwise"

and the set E^M.

claim 1: $C \subseteq E^M$

proof: Assume $x \in C$. Since $f_n(x) - f(x) \to 0$,

$\exists N_M$ such that $n \geq N_M \Rightarrow |f_n(x) - f(x)| < 1/M$.

i.e. $x \in E^M_{N_M} \Rightarrow x \in \bigcup_{n=1}^{\infty} E^M_n = E^M$, as desired.

claim 2: It's not necessarily true that $C = E^M$

proof: Define f_n on $[0,1]$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x \neq 1/2 \\ \frac{1}{10}\sin(n) & \text{if } x = 1/2 \end{cases}$$

Then $f_n \to 0$ at all points except $x = 1/2$. i.e.

$C = [0,1] - \{1/2\}$. But $\{|f_i - 0| < 1/5\} = [0,1]$

$\Rightarrow E^5_n = \bigcap_{i \geq n}^{\infty} \{|f_i - 0| < 1/5\} = [0,1]$

$\Rightarrow E^5 = \bigcup_{n=1}^{\infty} E^5_n = [0,1] \supsetneq C$, as claimed.

<u>claim 3:</u> $C = \bigcap_{M=1}^{\infty} E^M$

<u>proof:</u> assume $x \in C$. want $x \in E^M$ for all $M \geq 1$

Fix M. then since $x \in C$ we know $\exists N_M$ such that $|f_i(x) - f(x)| < \frac{1}{M}$ for all $i \geq N_M$. Hence $x \in E^M_{N_M} \Rightarrow x \in \bigcup_{n=1}^{\infty} E^M_n = E^M$. Since M was arbitrary, $x \in \bigcap_{M=1}^{\infty} E^M$ as desired. $\Rightarrow C \subseteq \bigcap_{M=1}^{\infty} E^M$

Assume $x \in \bigcap_{M=1}^{\infty} E^M$. Want to show $x \in C$.

Fix $\varepsilon > 0$. Choose M so that $\frac{1}{M} < \varepsilon$. We know $x \in E^M = \bigcup_{n=1}^{\infty} E^M_n \Rightarrow x \in E^M_{\tilde{n}}$ for some $\tilde{n}$.

$\Rightarrow |f_n(x) - f(x)| < \frac{1}{M} < \varepsilon$ for all $n > \tilde{n}$. This proves that $f_n(x) \to f(x)$ and so $x \in C$.

$\Rightarrow \bigcap_{M=1}^{\infty} E^M \subseteq C.$ //