Here are the questions from the April 2001 Final Exam in MAT 187H1S, with answers. 374 students wrote this exam. The marks ranged from 7% to 93%, with an average of (only) 54.1%

1. (15 marks: avg: 11.7) Find the following:

   (a) (5 marks) $\int \frac{\cot(\ln x)}{x}\,dx$

   (b) (5 marks) the length of the curve with parametric equations $x = 4\cos t\,; y = 4\sin t\,; z = 3t$ for $0 \le t \le 2\pi$.

   (c) (5 marks) $\frac{\partial^2}{\partial x \partial y}\left(\sqrt{1+xy^2}\right)$ at the point $(x, y) = (1, 1)$

2. (15 marks; avg: 9.2) Find the general solution to each of the following differential equations:

   (a) (6 marks) $\frac{dy}{dx} = 2x(y^2+4)$

   (b) (9 marks) $\frac{dy}{dx} + \frac{2}{x}y = \frac{\sin x}{x}$

3. (15 marks; avg: 7.7) Let $f(x) = \sum_{n=0}^{\infty}(-4)^n \frac{n+1}{n^2+1}x^{2n} = 1 - 4x^2 + \frac{48}{5}x^4 - \frac{128}{5}x^6 + \frac{1280}{17}x^8 - \ldots$.

   Find the following:

   (a) (2 marks) $f^{(2)}(0)$

   (b) (2 marks) the 6*th* degree Taylor polynomial of $f(x)$ about $x = 0$

   (c) (2 marks) $\lim_{x\to 0}\frac{f(x)-1+4x^2}{x^4}$

   (d) (4 marks) the radius of convergence for $f(x)$

   (e) (5 marks) $\int_0^{0.5} x^2 f(x)\,dx$ correct to within .01

4. (15 marks; avg: 10.6) Consider the curve in the x-y plane with equation $x^2 - 6x + y^2 = 0$. Find the following:

   (a) (5 marks) the polar equation of the curve

   (b) (5 marks) the length of the curve

   (c) (5 marks) the area of the region within the curve

5. (10 marks; avg: 6.1) Find the critical points of $f(x, y) = 3x^4 - 6xy^2 - 4y^3$ and at each critical point determine whether f has a relative maximum point, a relative minimum point, or a saddle point.

6. (10 marks; avg: 3.3) Do the following infinite series converge or diverge? Justify your answer.

(a) (3 marks) $\sum_{n=1}^{\infty} \frac{n+2}{n^3 + \sin n}$

(b) (3 marks) $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{1+n^2}}$

(c) (4 marks) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

7. (10 marks; avg: 3.2) Find the following:

(a) (5 marks) the exact sum (*not* a decimal approximation) of $\sum_{n=1}^{\infty} \frac{n}{4^n}$.

(b) (5 marks) the first four (non-zero) terms of the Maclaurin series of $f(x) = \frac{3x+7}{x^2+2x-15}$. What is the radius of convergence for this series?

8. (10 marks; avg: 2.5) Find $\int_0^{\infty} \frac{x}{(x+3)(x^2-3x+9)}\,dx$.

ANSWERS: 1(a) $= \ln|\csc(\ln x)| + c$ (b) 10π (c) $\frac{3}{4\sqrt{2}}$

2.(a) $y = 2\tan(2x^2 + c)$ (b) $y = -\frac{\cos x}{x} + \frac{\sin x}{x^2} + \frac{c}{x^2}$

3.(a) -8 (b) $1 - 4x^2 + \frac{48}{5}x^4 - \frac{128}{5}x^6$ (c) $\frac{48}{5}$ (d) $R = 1/2$ (e) $.0273808$

4.(a) $r = 6\cos\theta$ (b) 6π (c) 9π

5. $(0,0)$ is a saddle point; $(1/2, -1/2)$ is a relative minimum point

6.(a) converges (b) diverges (c) converges

7.(a) $\frac{4}{9}$ 7(b) $-\frac{7}{15} - \frac{59}{225}x - \frac{223}{3375}x^2 - \frac{1331}{50625}x^3$; $R = 3$

8. First, we use partial fractions as follows:

$$\int_0^{\infty} \frac{x}{(x+3)(x^2-3x+9)}\,dx = \frac{1}{9}\int_0^{\infty} -\frac{1}{x+3} + \frac{x+3}{x^2-3x+9}\,dx$$

Now you have to be very very careful. If you try to break it up as two integrals

$$\frac{1}{9}\int_0^{\infty} -\frac{1}{x+3}\,dx + \frac{1}{9}\int_0^{\infty} \frac{x+3}{x^2-3x+9}\,dx$$

Then the first integral diverges to $-\infty$ and the second one diverges to ∞. In fact, the second integral has an integrand that decays like $1/x$ as x gets large. So this suggests that there might be a nice cancellation to help us out. *So we don't want to break up the integrals.* In fact, this understanding tells us that we want to do a substitution: $u = x+3$. Doing this, we get the integral

$$\frac{1}{9}\int_3^\infty -\frac{1}{u} + \frac{u}{u^2 - 9u + 27}\, du.$$

I know that the second term is like $1/u$ for large u so I'm going to add and subtract a term to allow me to extract this behavior.

$$\frac{1}{9}\int_3^\infty -\frac{1}{u} + \frac{u - 9/2}{u^2 - 9u + 27} + \frac{9/2}{u^2 - 9u + 27}\, du.$$

We know the antiderivative of the first two terms:

$$\int -\frac{1}{u} + \frac{u - 9/2}{u^2 - 9u + 27}\, du = -\log(u) + \frac{1}{2}\log(u^2 - 9u + 27) = \log\left(\frac{\sqrt{u^2 - 9u + 27}}{u}\right)$$

So we can take care of the first two terms in the integrand. In fact

$$0 = \frac{1}{9}\int_3^\infty -\frac{1}{u} + \frac{u - 9/2}{u^2 - 9u + 27}\, du.$$

So we're almost done. We've figured out that

$$\int_0^\infty \frac{x}{(x+3)(x^2 - 3x + 9)}\, dx = \frac{1}{9}\int_3^\infty \frac{9/2}{u^2 - 9u + 27}\, du.$$

We can do this integral by trigonometric substitution. It equals

$$\frac{2\pi}{9\sqrt{3}}.$$