Mat 135 Sept 27, 2004

§ 2.5 Continuity.

A function being continuous at a point a is stronger than the function's simply having a limit at a.

definition: f is continuous at a if

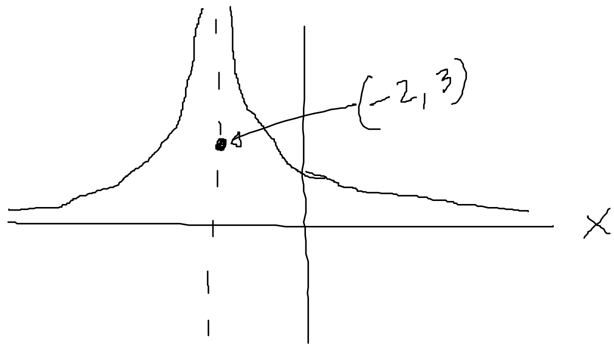
1) $f(a)$ is defined
2) $\lim_{x \to a} f(x)$ exists
3) $\lim_{x \to a} f(x) = f(a)$.

Being continuous at a is stronger than having a limit at a because there are two additional requirements in order to be continuous at a.

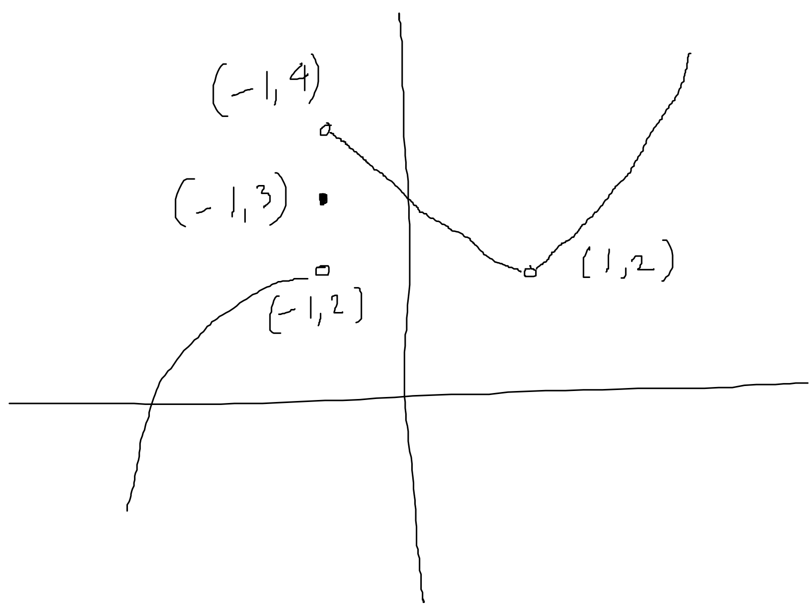
ex: $f(x) = \frac{1}{(x+2)^2}$ is not continuous at $a = -2$. why? $f(-2)$ isn't defined!

ex: $f(x) = \begin{cases} \frac{1}{(x+2)^2} & x \neq -2 \\ 3 & x = -2 \end{cases}$ is not continuous at $a = -2$.

why? $\lim_{x \to -2} f(x) \neq f(a)$



$$f(x) = \begin{cases} x^2+1 & x > 1 \\ -x+3 & -1 < x < 1 \\ 3 & x = -1 \\ -(x+1)^2+2 & x < -1 \end{cases}$$

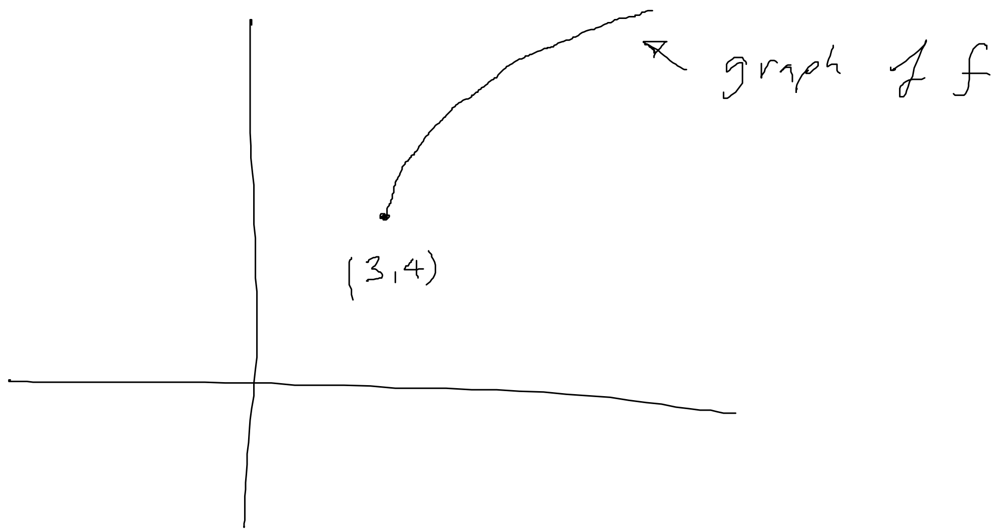


f is not cont. at 1 since $f(1)$ is not defined.

f is not cont. at -1 since $\lim_{x \to -1} f(x)$ does not exist.

f is continuous at 0.

$f(x) = \sqrt{x-3} + 4$



Note that $\lim_{x \to 3} f(x)$ does not exist. Why? To have a limit at 3, you have to let x approach 3 from both sides. For this function, f is not defined if $x < 3$.

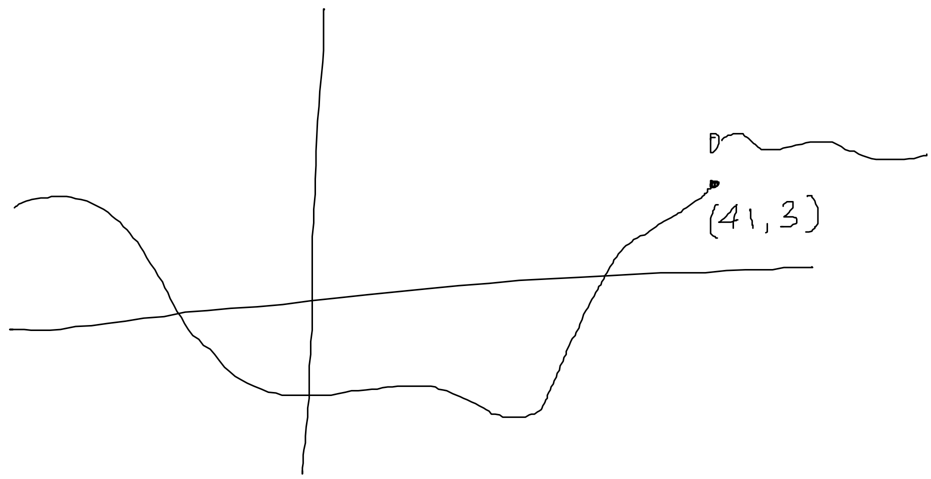
$\Rightarrow \lim_{x \to 3^+} f(x)$ exists and equals 4.

$f(3)$ is defined and equals 4.

$\lim_{x \to 3^+} f(x) = f(3)$

In this case, we say that "f is continuous from the right at 3"

If f is a function whose graph is



then f is continuous from the left at 41 but is not continuous from the right at 41

<u>theorem</u>: if f and g are continuous at a and c is a constant then the following functions are also continuous at a:

1. $f+g$ 2. $f-g$ 3. cf
4. fg 5. f/g if $g(a) \neq 0$

this is very natural from the limit laws of section 2.3

For example,

$$f(x) = \frac{x^2+1}{x-2}$$

is continuous at every point in $\mathbb{R}$ except $x=2$

Why?

First of all, we check that the functions

$g(x) = x$ and $h(x) = 1$ are

continuous at every point in $\mathbb{R}$. Then, we know that g^2 is continuous at every point in $\mathbb{R}$. (by 4.) And then we know that g^2+h is continuous at every point in $\mathbb{R}$ (by 1.) So the function $n(x) = x^2+1$ is continuous at every point in $\mathbb{R}$. Similarly, $d(x) = x-2$ is continuous at every point in $\mathbb{R}$. And so by 5, n/d is continuous at every point of $\mathbb{R}$ where $d \neq 0$. Since $d=0$ at only on point ($d(2)=0$), we have n/d continuous on $(-\infty, 2) \cup (2, \infty)$ as claimed.

<u>theorem</u>: The following types of functions are continuous at every point in their domain: polynomials, rational functions, root functions, trig funct., inverse trig. functions, exponential functions, logarithmic functions.

ex:
$$f(x) = \frac{\sqrt{x+1} - \ln(-x)}{x + 1/2}$$

Where is f continuous? First, where is f defined?

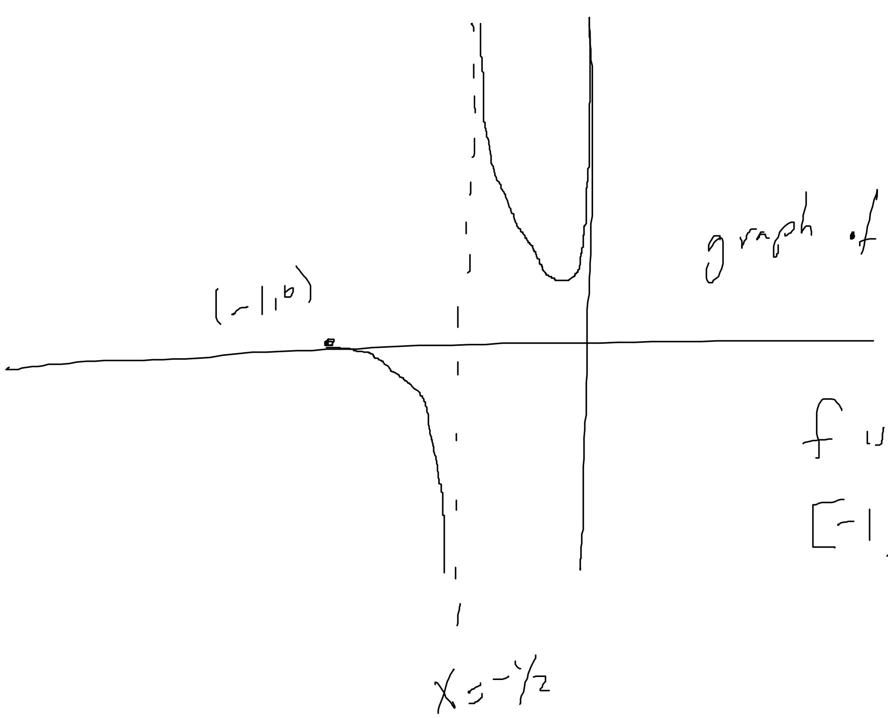
$\sqrt{x+1}$ is defined if $x \geq -1$

$\ln(-x)$ is defined if $x < 0$

$\frac{1}{x+1/2}$ is defined if $x \neq -1/2$

So f(x) is defined if

$-1 \leq x < -1/2$ or $-1/2 < x < 0$.

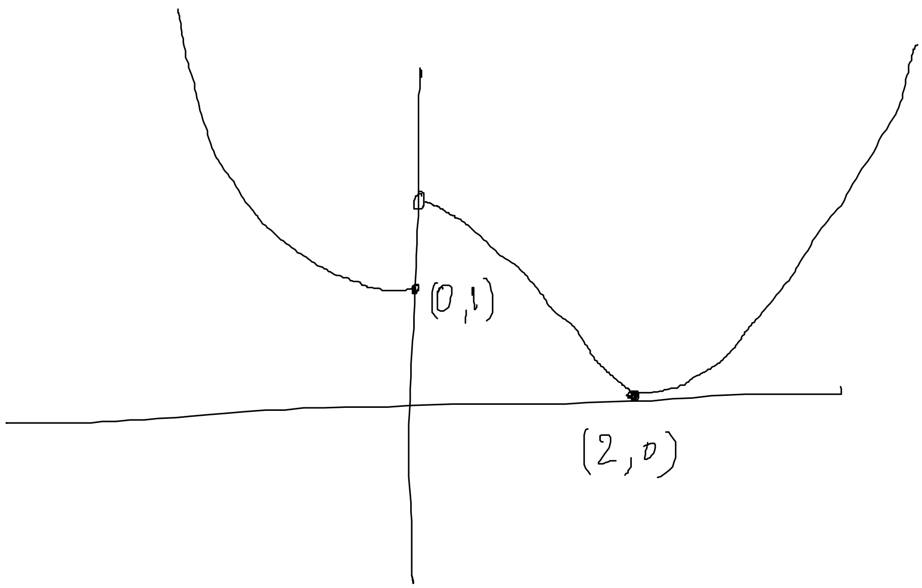


graph of f.

f is continuous on $[-1, -1/2) \cup (-1/2, 0)$

ex 37.

$$f(x) = \begin{cases} 1+x^2 & x \le 0 \\ 2-x & 0 < x \le 2 \\ (x-2)^2 & x > 2 \end{cases}$$



f is continuous at all x except x = 0.

f is continuous from the left at 0.

ex 36:

$$f(x) = \begin{cases} \sin x & x < \pi/4 \\ \cos x & x \ge \pi/4 \end{cases}$$

f is continuous at all points in $\mathbb{R}$