Mat 135. Sept 24, 2004

§2.3 calculating limits using the limit laws

We can't <u>prove</u> what limits are in this course, but we can provide you with some rules that will give you more security in stating what you believe the limit is

<u>rule</u>

1) if $f(x) = c$ is a constant function then $\lim_{x \to a} f(x) = c$.

2) if $f(x) = x$ then

$$\lim_{x \to a} f(x) = a.$$

Using these two facts and the following rules we can calculate many limits.

<u>Limit laws</u> Suppose c is a constant and the following limits exist:

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

then

1) $\lim_{x\to a}(f(x)+g(x)) = \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a} g(x)\right)$

3) $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$

4) $\lim_{x\to a}(f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$

5) $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

---

ex: $\lim_{x\to 2} 3x+4x^2-9 = 13$ why?

we know $\lim_{x\to 2}(-9) = -9$ & $\lim_{x\to 2} x = 2$.

$\Rightarrow \lim_{x\to 2} 3x = 3\lim_{x\to 2} x$ by rule 3

$= 3\cdot 2 = 6$

$\lim_{x\to 2} x^2 = \left(\lim_{x\to 2} x\right)\left(\lim_{x\to 2} x\right)$ by rule 4

$= 2\cdot 2 = 4$

$\Rightarrow \lim_{x\to 2} 4x^2 = 4\lim_{x\to 2} x^2$ by rule 3

$= 16$

So $\lim\limits_{x\to 2}(3x+4x^2) = \lim\limits_{x\to 2} 3x + \lim\limits_{x\to 2} 4x^2$ by rule 1

$= 6 + 16 = 22$

So $\lim\limits_{x\to 2}(3x+4x^2-9) = \lim\limits_{x\to 2}(3x+4x^2) + \lim\limits_{x\to 2}(-9)$ by rule 1

$= 22 - 9 = 13.$

And so, by using the various limit rules we found

$$\lim_{x\to 2} 3x+4x^2-9 = 13, \text{ as claimed.}$$

<u>Note</u>: by applying the product law many times, we see that

$$\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n \text{ for any positive integer } n$$

if $\lim\limits_{x\to a} f(x)$ exists.

it follows that $\lim\limits_{x\to a} x^n = a^n$ for any positive integer

Also, one has a root law:

if $\lim\limits_{x\to a} f(x) \geq 0$ and n is an even positive integer then $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$

ex: find $\lim\limits_{h\to 0} \dfrac{(2+h)^3-8}{h}$

since the denominator goes to 0 as $h\to 0$, we can't just say

$$\frac{\lim\limits_{h\to 0} (2+h)^3-8}{\lim\limits_{h\to 0} h}$$

On the other hand, we see that the numerator goes to 0 as $h\to 0$ so we'll try to see if there's some cancellation to help us out.

$$\lim_{h\to 0} \frac{(2+h)^3-8}{h} = \lim_{h\to 0} \frac{(h^3+6h^2+12h+8)-8}{h}$$

$$= \lim_{h\to 0} \frac{h^3+6h^2+12h}{h}$$

$$= \lim_{h\to 0} h^2+6h+12 = 12$$

ex: $\lim_{x \to -4} \dfrac{x^2+5x+4}{x^2+3x-4}$

as $x \to -4$, the numerator goes to 0 and the denominator goes to 0. Since the numerator and denominator are polynomials, this means that $(x-(-4))$ must be a factor of each of them:

$$x^2+5x+4 = (x+4)(x+1)$$
$$x^2+3x-4 = (x+4)(x-1)$$

$$\Rightarrow \lim_{x \to -4} \frac{x^2+5x+4}{x^2+3x-4} = \lim_{x \to -4} \frac{(x+4)(x+1)}{(x+4)(x-1)}$$

$$= \lim_{x \to -4} \frac{x+1}{x-1}$$ ← can apply limit law to this

$$= \frac{-3}{-5} = \frac{3}{5}$$

ex: $\lim_{x \to 2} \dfrac{|x-2|}{x-2}$

absolute values are tricky, we'll do this problem by computing the limits from the left & from the right.

$$\lim_{x \to 2^+} \frac{|x-2|}{x-2} = \lim_{x \to 2+} \frac{(x-2)}{x-2}$$

since $x-2>0$ and so $|x-2| = x-2$

$$= \lim_{x \to 2+} 1 = 1$$

$$\lim_{x \to 2-} \frac{|x-2|}{x-2} = \lim_{x \to 2-} \frac{-(x-2)}{x-2}$$

since $x-2<0$ and so $|x-2| = -(x-2)$

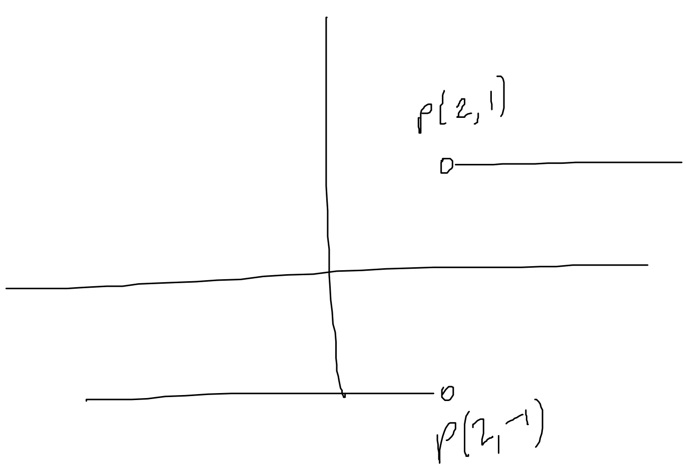
$$= \lim_{x \to 2^-} -1 = -1$$

Since $\displaystyle\lim_{x \to 2^+} \frac{|x-2|}{x-2} \neq \lim_{x \to 2^-} \frac{|x-2|}{x-2}$ we know

that $\displaystyle\lim_{x \to 2} \frac{|x-2|}{x-2}$ doesn't exist.

Not surprising, since

$$f(x) = \frac{|x-2|}{x-2} = \begin{cases} 1 & \text{if } x>2 \\ -1 & \text{if } x<2 \end{cases}$$



graph of f.