Sept 22, 2004

Can we reliably use calculators to find limits?
No! You have to be very careful.

Imagine trying to find

$$\lim_{x \to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x}$$

using a calculator that has only 8 significant digits. Such a calculator will give you

x	$f(x)$
1	0.2679492
1/10	0.2683090
1/100	0.2884400
1/1000	0.2887000
1/10,000	0.2890000
1/100,000	0.2900000
1/1,000,000	0.3000000
1/10,000,000	0

does this mean

$$\lim_{x \to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x} = 0 \ ?$$

No! The limit is $\sqrt{3}/6$.

<u>Q1</u>: what happended w/ the calculator?

$$x = 1/10{,}000{,}000 \Rightarrow \sqrt{x+3} \cong 1.7320508$$

$$\sqrt{3} \cong 1.7320508$$

So up to 8 significant digits,

$\sqrt{x+3} - \sqrt{3} = 0$ for

$$x = 1/10{,}000{,}000$$

They disagree at higher significant digits, but the calculator's blind to those

<u>Q2</u>: How do you know the limit is $\sqrt{3}/6$?

<u>A</u>: Multiply numerator & denominator by $\sqrt{x+3} + \sqrt{3}$

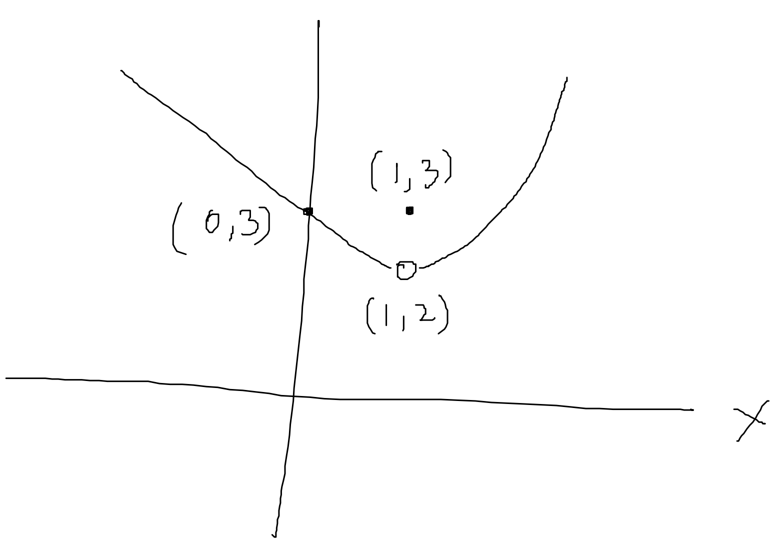
$$\frac{\sqrt{x+3}-\sqrt{3}}{x} \cdot \frac{\sqrt{x+3}+\sqrt{3}}{\sqrt{x+3}+\sqrt{3}}$$

$$= \frac{(x+3)-3}{x(\sqrt{x+3}+\sqrt{3})} = \frac{x}{x(\sqrt{x+3}+\sqrt{3})}$$

So $\lim\limits_{x \to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x} = \lim\limits_{x \to 0} \frac{1}{\sqrt{x+3}+\sqrt{3}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}.$

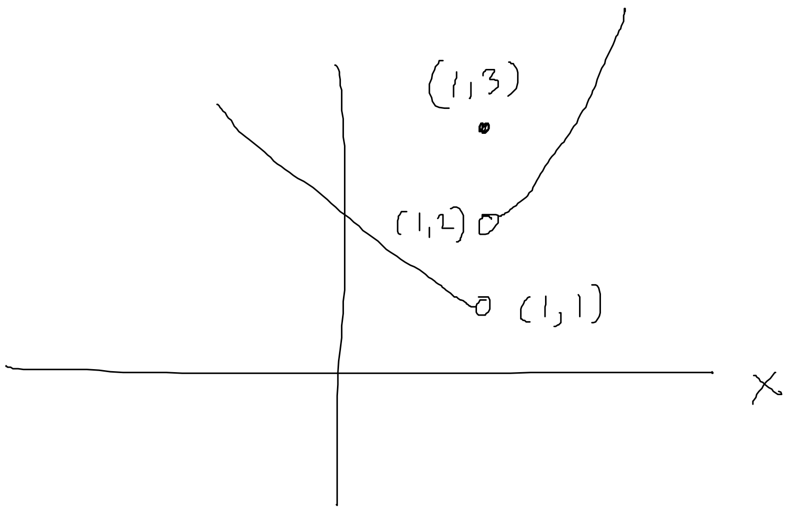
One sided limits

$$f(x) = \begin{cases} x^2+1 & x > 1 \\ 3 & x = 1 \\ -x+3 & x < 1 \end{cases}$$



$\lim\limits_{x \to 1} f(x) = 2$

$$f(x) = \begin{cases} x^2+1 & x>1 \\ 3 & x=1 \\ -x+2 & x<1 \end{cases}$$



$\lim_{x\to 1} f(x)$ does not exist

$\lim_{x\to 1+} f(x) = 2$

$\lim_{x\to 1-} f(x) = 1$

<u>defn</u>: We write $\lim_{x\to a-} f(x) = L$ and say "the left-hand limit of f as x approaches a is equal to L" if we can make the values of f arbitrarily close to L by taking x to be sufficiently close to a and less than a.

$\lim_{x \to a+} f(x) = L$ is defined analogously.

<u>fact:</u> $\lim_{x \to a} f(x) = L$ if and only if

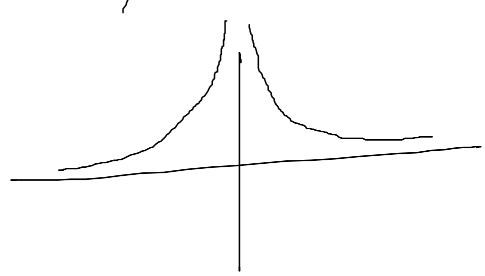
$\lim_{x \to a-} f(x) = L$ and $\lim_{x \to a+} f(x) = L$.

<u>defn:</u> Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.

<u>ex:</u> $f(x) = \frac{1}{x^2}$



then $\lim_{x \to 0} f(x) = \infty$.

How do you see that

$$\lim_{x \to 0} f(x) = \infty \,?$$

x	$f(x)$	x	$f(x)$
$0+1$	1	$0-1$	1
$0+1/10$	100	$0-1/10$	100
$0+1/100$	$10{,}000$	$0-1/100$	$10{,}000$
$0+1/1000$	$1{,}000{,}000$	$0-1/1000$	$1{,}000{,}000$
$0+1/10000$	$100{,}000{,}000$	$0-1/10000$	$100{,}000{,}000$

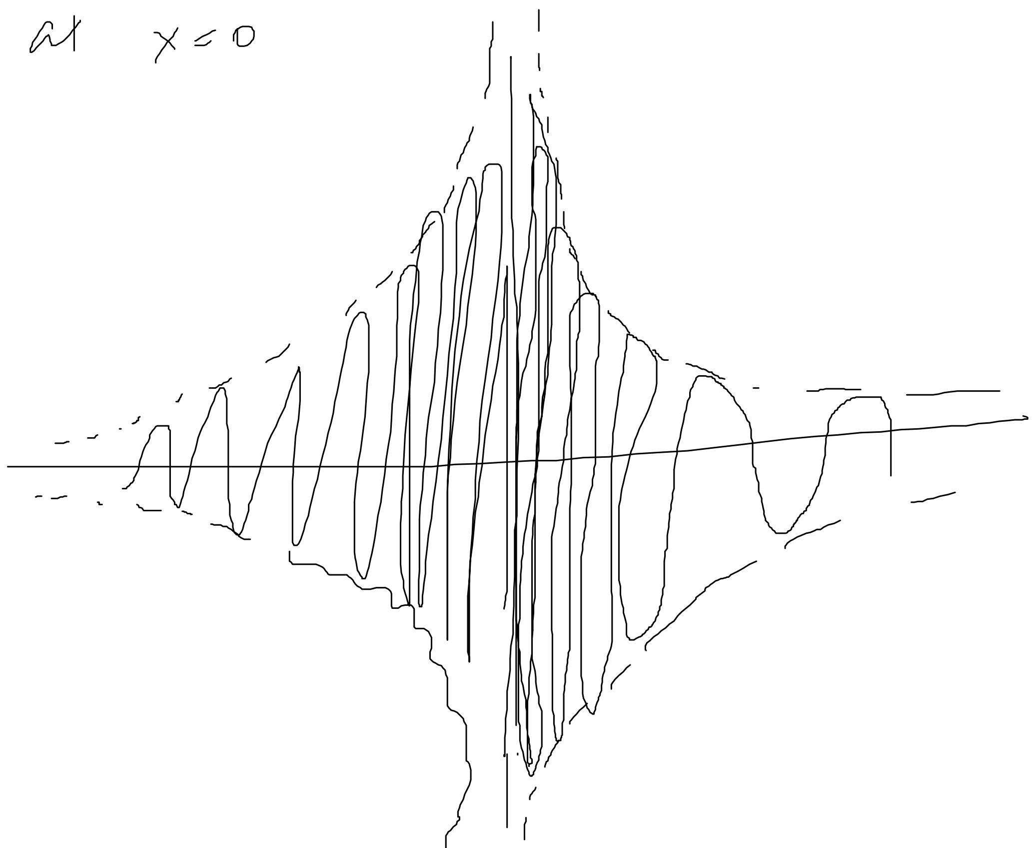
The closer x is to 0, the larger $f(x)$ is.

Given M, $f(x) > M$ if $|x| < 1/\sqrt{M}$.

So you can ensure that $f(x)$ is large by taking x close to 0. No matter how large you want $f(x)$ to be, it can be guaranteed to be true for all x close enough to 0.

Note.

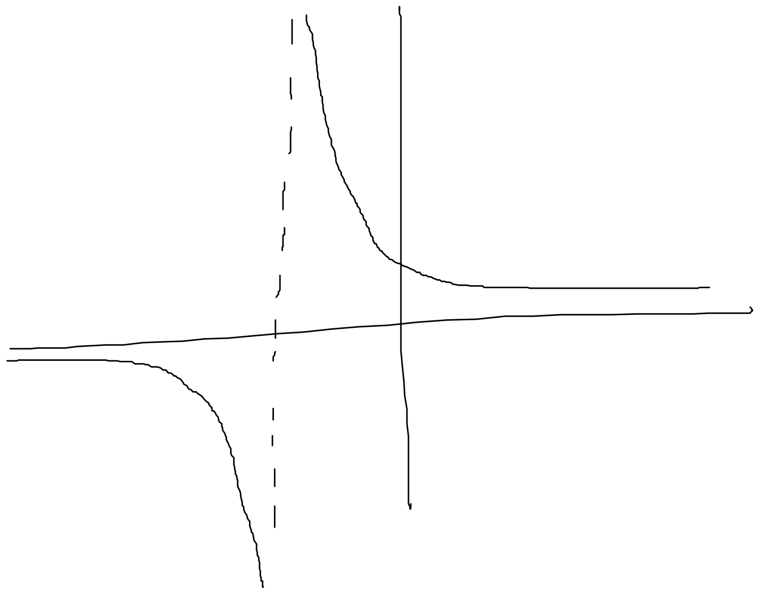
$f(x) = \frac{1}{x^2} \sin\left(\frac{1}{x}\right)$ does not have a limit at $x = 0$



even though there are x close to 0 so that $f(x)$ is really large, for $\lim_{x \to 0} f(x) = \infty$ we need that given M we can find an interval containing 0 so that $f(x) > M$ for all x in that interval. Not just for some.

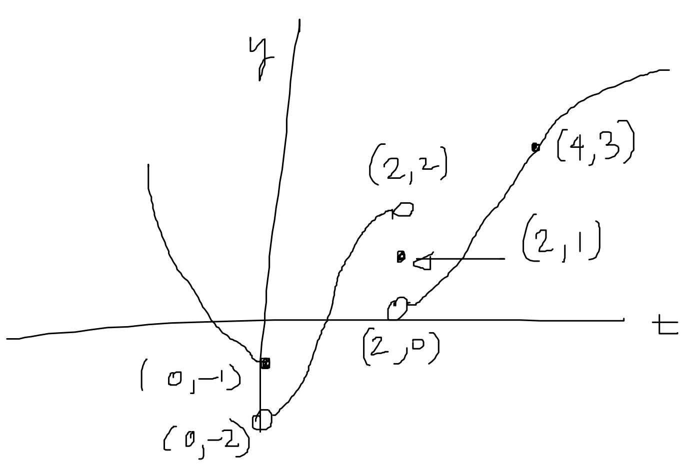
$\lim_{x \to -2} \frac{1}{x+2}$ does not exist

but



$\lim_{x \to (-2)+} \frac{1}{x+2} = \infty$ and $\lim_{x \to (-2)-} \frac{1}{x+2} = -\infty$.

ex 7: for the function whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.



$\lim_{t \to 0-} g(t) = -1$

$\lim_{t \to 0+} g(t) = -2$

$\lim_{t \to 0} g(t)$ doesn't exist since left-hand limit $\neq$ right-hand limit

$$\lim_{t \to 2^-} g(t) = 2$$

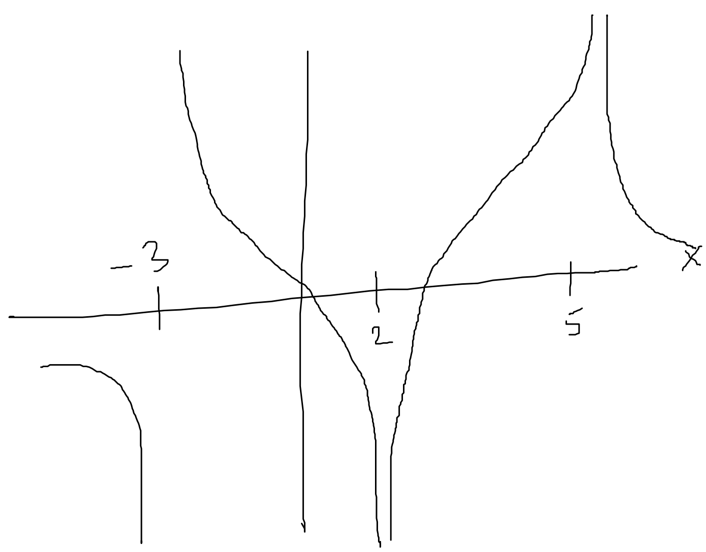
$$\lim_{t \to 2^+} g(t) = 0$$

$\lim_{t \to 2} g(t)$ does not exist, since left-hand limit $\neq$ right-hand limit

$$g(2) = 1$$

$$\lim_{t \to 4} g(t) = 3$$

8 for the function whose graph is shown state the following



$$\lim_{x \to 2} R(x) = -\infty$$

$$\lim_{x \to 5} R(x) = \infty$$

$$\lim_{x \to (-3)^-} R(x) = -\infty$$

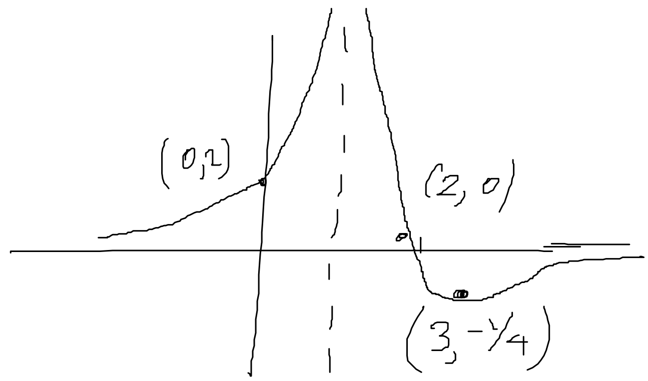
$$\lim_{x \to (-3)^+} R(x) = \infty$$

vertical asymptotes are

$x = -3, \ x = 2, \ x = 5$

ex: plot graph & find all interesting limits.

$$f(x) = \frac{2-x}{(x-1)^2}$$



num>0 den>0		num>0 den>0		num<0 den>0
	1		2	
$f(x)>0$		$f(x)>0$		$f(x)<0$

$$\lim_{x\to 1+} f(x) = \infty$$

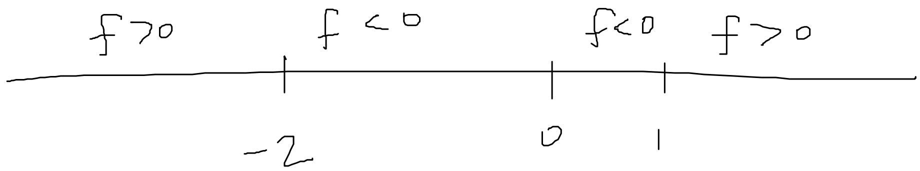
$$\lim_{x\to 1-} f(x) = \infty$$

$$\lim_{x\to 1} f(x) = \infty$$

ex: same question, $f(x) = \dfrac{x-1}{x^2(x+2)}$

num<0 den<0		num<0 den>0		num<0 den>0		num>0 den>0
	-2		0		1	

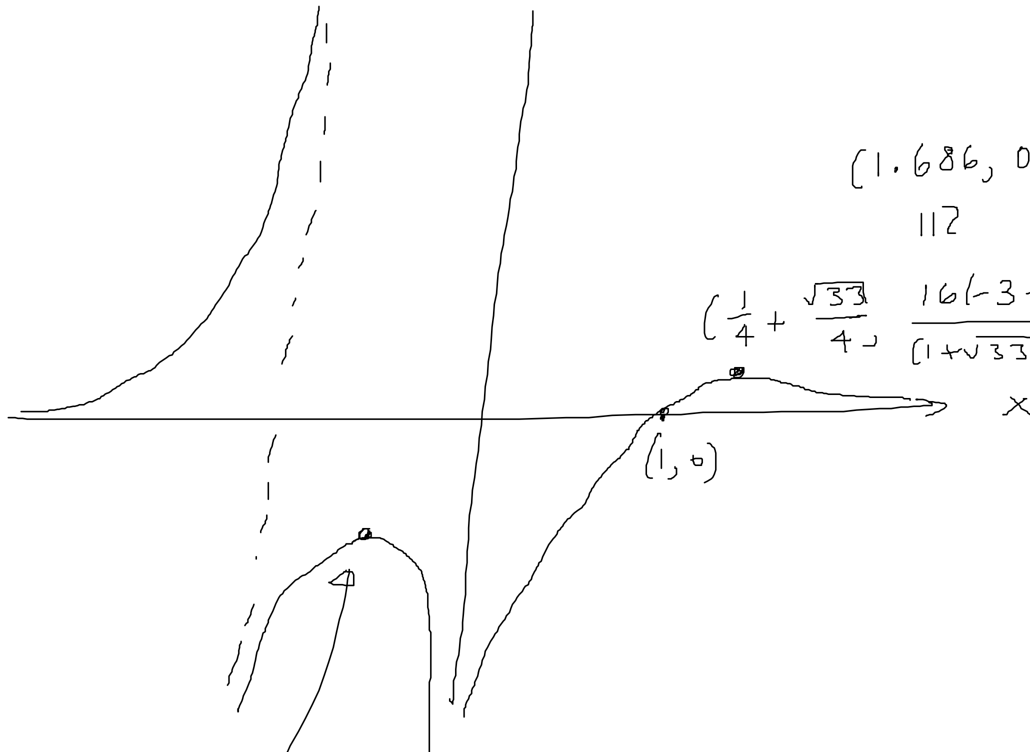
So



$(1.686, 0.0654)$

$\cong$

$\left(\frac{1}{4} + \frac{\sqrt{33}}{4}, \frac{16(-3+\sqrt{33})}{(1+\sqrt{33})^2(9+\sqrt{33})}\right)$



$$\left(\frac{1}{4} - \frac{\sqrt{33}}{4}, \frac{16(3+\sqrt{33})}{(-1+\sqrt{33})^2(-9+\sqrt{33})}\right)$$

$$\cong (-1.186, -1.909)$$

$\lim_{x \to (-2)^-} f(x) = \infty$

$\lim_{x \to 0} f(x) = -\infty$

$\lim_{x \to (-2)^+} f(x) = -\infty$

$\lim_{x \to 1} f(x) = 0$

$\lim_{x \to -2} f(x)$ doesn't exist