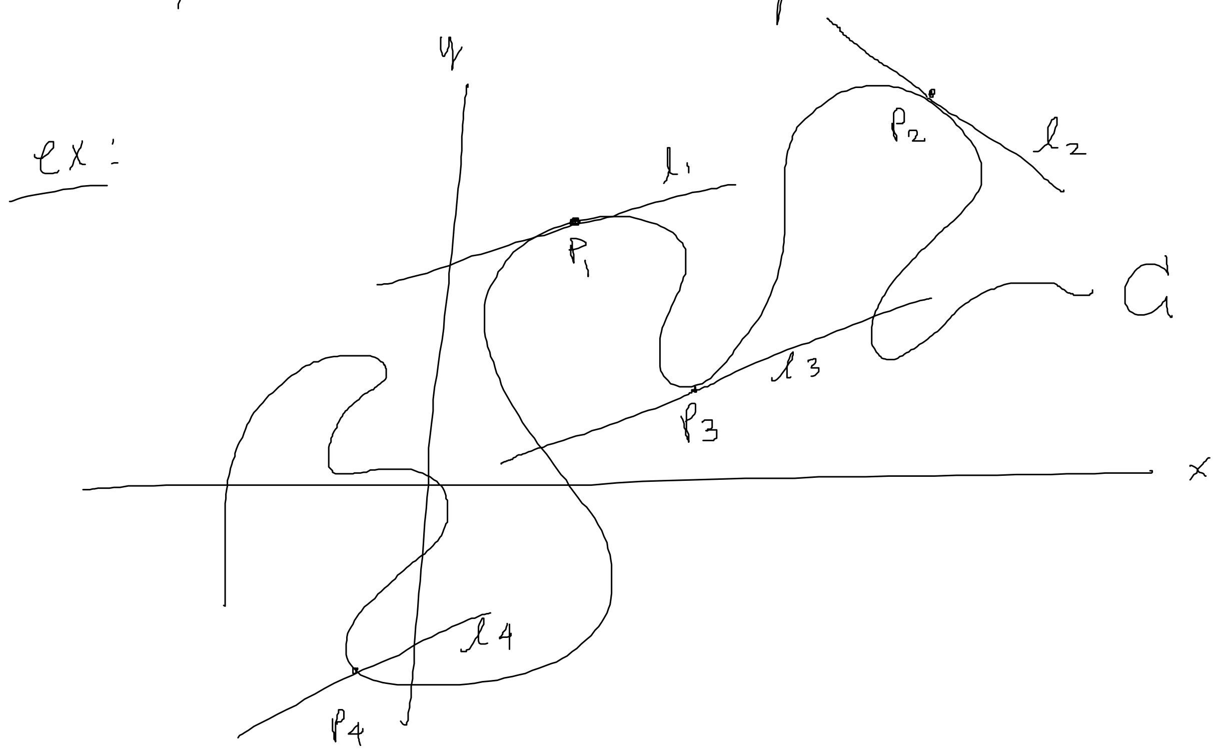
Sept 20, 2004

§2.1 The tangent and velocity problems.

Given a curve and a point on the curve, the tangent line to the curve at that point is a line that touches the curve at that point and has the same direction as the curve at that point.

ex:



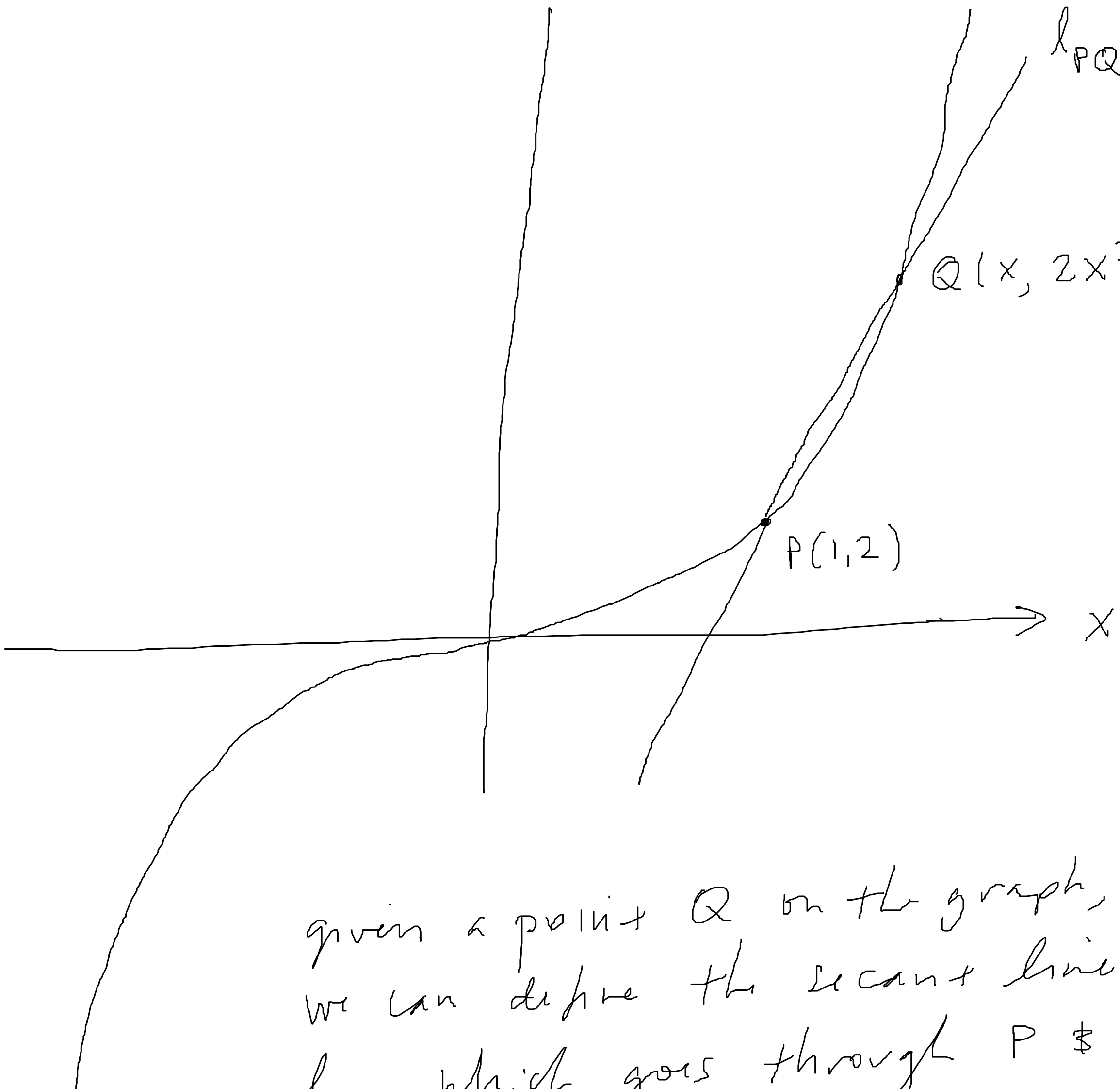
l_1 is tangent to C at P_1

l_2 is tangent to C at P_2

l_3 is tangent to C at P_3

l_4 is not tangent to C at P_4

Consider the graph of $f(x) = 2x^3$ at the point $P(1,2)$



given a point Q on the graph, we can define the secant line l_{PQ} which goes through P & Q and has slope m_{PQ}. Let's look at this for some points Q.

$$\boxed{l_{PQ} = m_{PQ}(x-1) + 2}$$

some secant lines where Q
is to the right of $P(1,2)$

	m_{PQ}	l_{PQ}
$Q(2, 16)$	14	$14(x-1)+2 = 14x - 12$
$Q(1\frac{1}{2}, \frac{27}{4})$	$19/2$	$\frac{19}{2}(x-1)+2 = \frac{19x}{2} - \frac{15}{2}$
$Q(1\frac{1}{4}, \frac{125}{32})$	$\frac{61}{8} = 7.625$	$\frac{61}{8}(x-1)+2 = \frac{61}{8}x - \frac{45}{8} = 7.625x - 5.625$
$Q(1\frac{1}{10}, \frac{1331}{500})$	$\frac{331}{50} = 6.62$	$\frac{331}{50}(x-1)+2 = 6.62x - 4.62$
$Q(1\frac{1}{100}, \frac{103031}{500000})$	$\frac{30301}{5000} = 6.0602$	$6.0602x - 4.0602$
$Q(1\frac{1}{1000}, \frac{1003003001}{500000100})$	6.006002	$6.006002x - 4.006002$

some secant lines where Q
is to the left of $P(1,2)$

	m_{PQ}	l_{PQ}
$Q(1-\frac{1}{10}, \frac{729}{500})$	5.42	$5.42x - 3.42$
$Q(1-\frac{1}{100}, \frac{970299}{500000})$	5.9402	$5.9402x - 3.9402$
$Q(1-\frac{1}{1000}, \frac{997002999}{500000000})$	5.994002	$5.994002x - 3.994002$

it seems that the closer the point Q gets to P(1, 2), the closer the secant line gets to the line

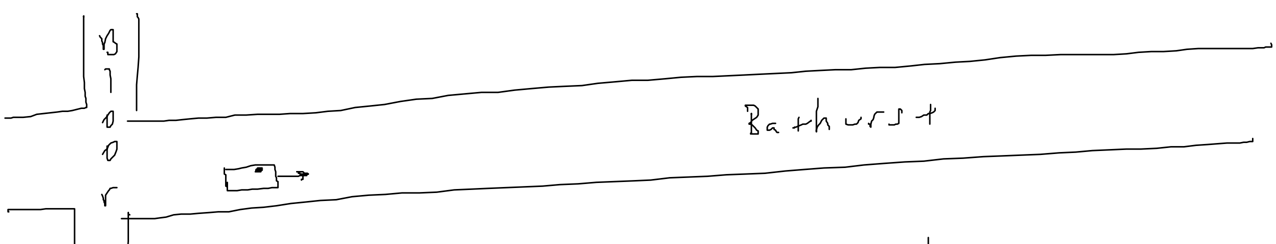
$$\boxed{6x-4}$$

In fact, $l(x) = 6x - 4$ is the tangent line to $f(x) = 2x^3$ at the point P(1, 2).

The slope of the tangent line is 6 and

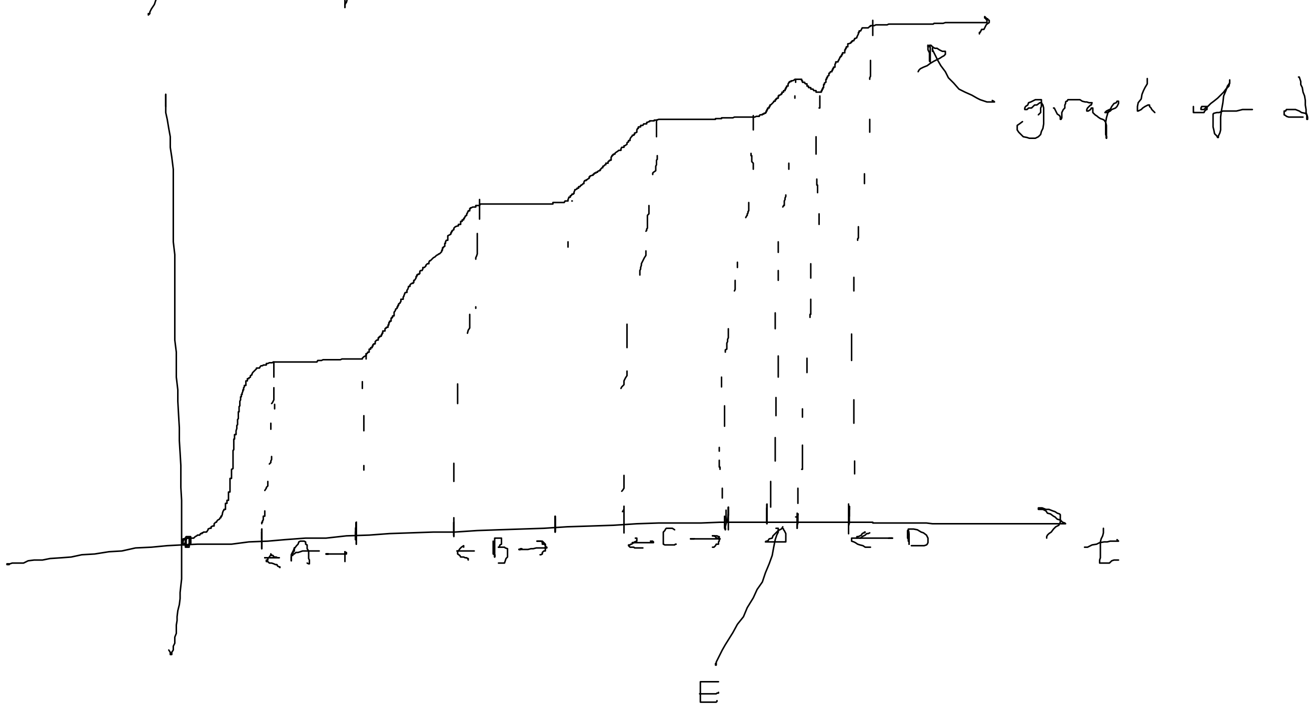
$$l(x) = 6(x-1) + 2 \quad ✓$$

Average velocities, instantaneous velocities.



You're in a car driving South on Bathurst. You started at Bloor at time 0 and as time advances, you find yourself $d(t)$ meters south of Bloor

Plotting your trajectory as a function of time, you might see something like this:



In time intervals A, B, C, and D you're stopped at traffic lights. In time interval E, you had to reverse to get around a stopped car. You drove fastest before time interval A and drove slowest between intervals B and C.

The <u>average velocity</u> at time t will be the secant line slope of $P(t, d(t))$ and $Q(\tilde{t}, d(\tilde{t}))$.

the instantaneous velocity will be the limit of the average velocities as Q gets closer to P.

to compute tangent lines and instantaneous velocities, we will need to understand limits.

§2.2 The limit of a function

Let's go back to $f(x) = 2x^3$. What are f's values as x gets closer and closer to 1?

x	$f(x)$
$1+1$	16
$1+1/10$	2.662
$1+1/100$	2.060602
$1+1/1000$	2.006006002
$1+1/10000$	2.000600060002
$1+1/100000$	2.000060000600002

x	$f(x)$
$1-1$	0
$1-1/10$	1.458
$1-1/100$	1.940598
$1-1/1000$	1.994005998
$1-1/10000$	1.999400059998
$1-1/100000$	1.999940000599998

It appears that we can make $f(x)$ as close to 2 as we want by taking x very close to 1. We express this by "the limit of the function $f(x) = 2x^3$ as x approaches 1 equals 2". The notation is

$$\lim_{x \to 1} 2x^3 = 2.$$

Note: In this case, $\lim_{x \to 1} 2x^3 = f(1)$

but it's not always true that

$$\lim_{x \to x_0} f(x) = f(x_0).$$

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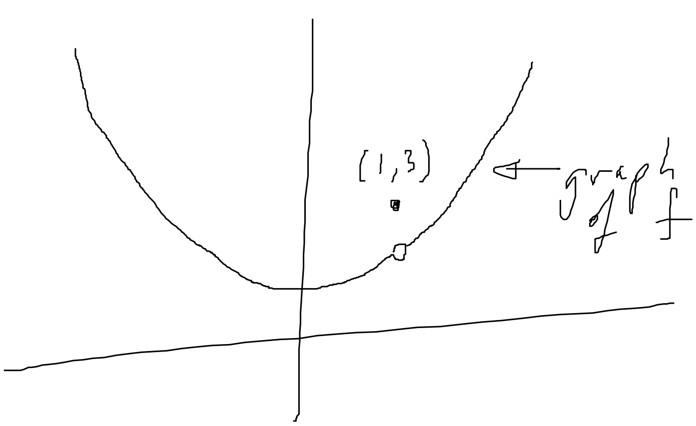
consider $\quad f(x) = \begin{cases} \dfrac{x^3 - x^2 + x - 1}{x-1} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$

$$= \begin{cases} x^2+1 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$$



x	$f(x)$
$1 + 1/10$	2.21
$1 + 1/100$	2.0201
$1 + 1/1000$	2.002001
$1 + 1/10000$	2.00020001
$1 + 1/100000$	2.0000200001

x	$f(x)$
$1 - 1/10$	1.81
$1 - 1/100$	1.9801
$1 - 1/1000$	1.998001
$1 - 1/10000$	1.99980001
$1 - 1/100000$	1.9999800001

$$\lim_{x \to 1} f(x) = 2 \neq f(1)$$

<u>Note</u>: we don't even need for $f(1)$ to be defined to look at $\lim_{x \to 1} f(x)$.

Not all functions have limits

$$f(x) = \begin{cases} \sin\left(\frac{1}{x-1}\right) & x \neq 1 \\ 0 & x = 1 \end{cases}$$

x	$f(x)$
$1 + 1/\pi$	0
$1 + 1/2\pi$	0
$1 + 1/3\pi$	0
$1 + 1/4\pi$	0
$1 + 1/10\pi$	0
$1 + 1/100\pi$	0
$1 + 1/1000\pi$	0

x	$f(x)$
$1 + 2/\pi$	1
$1 + 2/5\pi$	1
$1 + 2/9\pi$	1
$1 + 2/13\pi$	1
$1 + 2/41\pi$	1
$1 + 2/401\pi$	1
$1 + 2/40001\pi$	1

x	$f(x)$
$1 + 2/3\pi$	-1
$1 + 2/7\pi$	-1
$1 + 2/11\pi$	-1
$1 + 2/15\pi$	-1
$1 + 2/43\pi$	-1
$1 + 2/403\pi$	-1

I have 3 sequences of numbers such that $f(x) = 0$ on the first sequence, $f(x) = 1$ on the second sequence, $f(x) = -1$ on the third sequence.

$\lim_{x \to 1} f(x)$ does not exist!

Note: How did I find those sequences?

$f(x) = 0$ if

$\sin\left(\frac{1}{x-1}\right) = 0$ if

$\frac{1}{x-1} = n\pi$ for some integer n

$\Rightarrow x - 1 = 1/n\pi$

$\Rightarrow \boxed{x = 1 + 1/n\pi}$

$f(x) = 1$ if

$\sin\left(\frac{1}{x-1}\right) = 0$ if

$\frac{1}{x-1} = n2\pi + \pi/2$ for some integer n

$\Rightarrow \boxed{x = 1 + \frac{2}{n4\pi + \pi}}$

similarly,

$f(x) = -1$ if $\boxed{x = 1 + \frac{2}{n4\pi + 3\pi}}$ for some integer n

Q: is there some number such that I can get $f(x)$ as close as I want to that number, by taking x really close to 1?

A: No! No matter how close I take x to 1, $f(x)$ will achieve all values between -1 & 1. $\Rightarrow f(x)$ isn't getting close to a particular value