Mat 135 Sept 17, 2004

exponential functions

$x \to 5^x$ is an exponential function

$f(2) = 5^2 = 25$

$f(3) = 5^3 = 125$

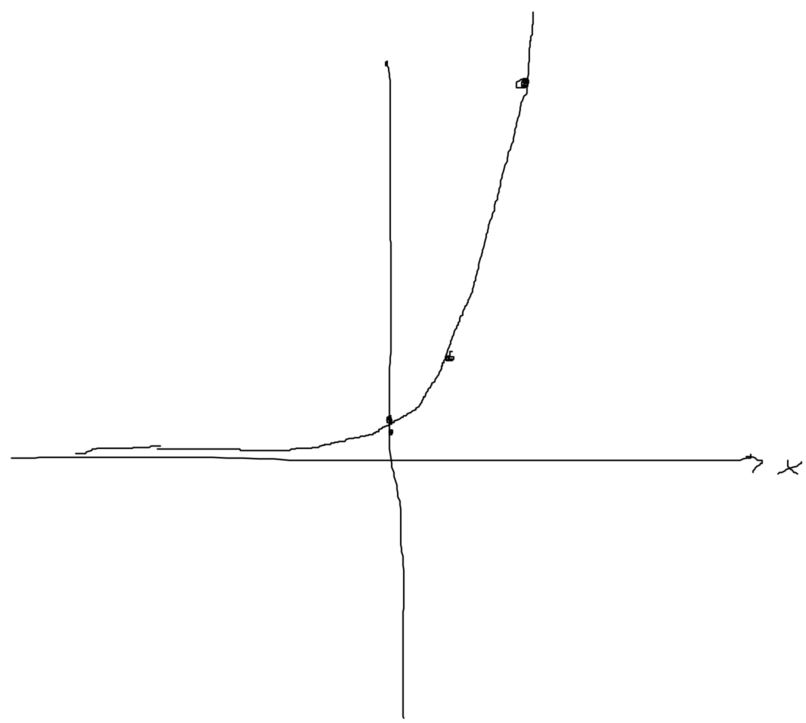
$f(1) = 5$

$f(0) = 1$

$f(-1) = 1/5$

$f(-2) = 1/25$

$f(-3) = 1/125$



can define f for rational values of x too

$f(4/9) = (5^4)^{1/9}$ for example

to define f for irrational values of x, we'll use "limits", which we haven't learnt yet. So you'll just have to believe me that the graph looks like I drew it.

Compare

$f(x) = 1^x$

$g(x) = 2^x$

$h(x) = 3^x$

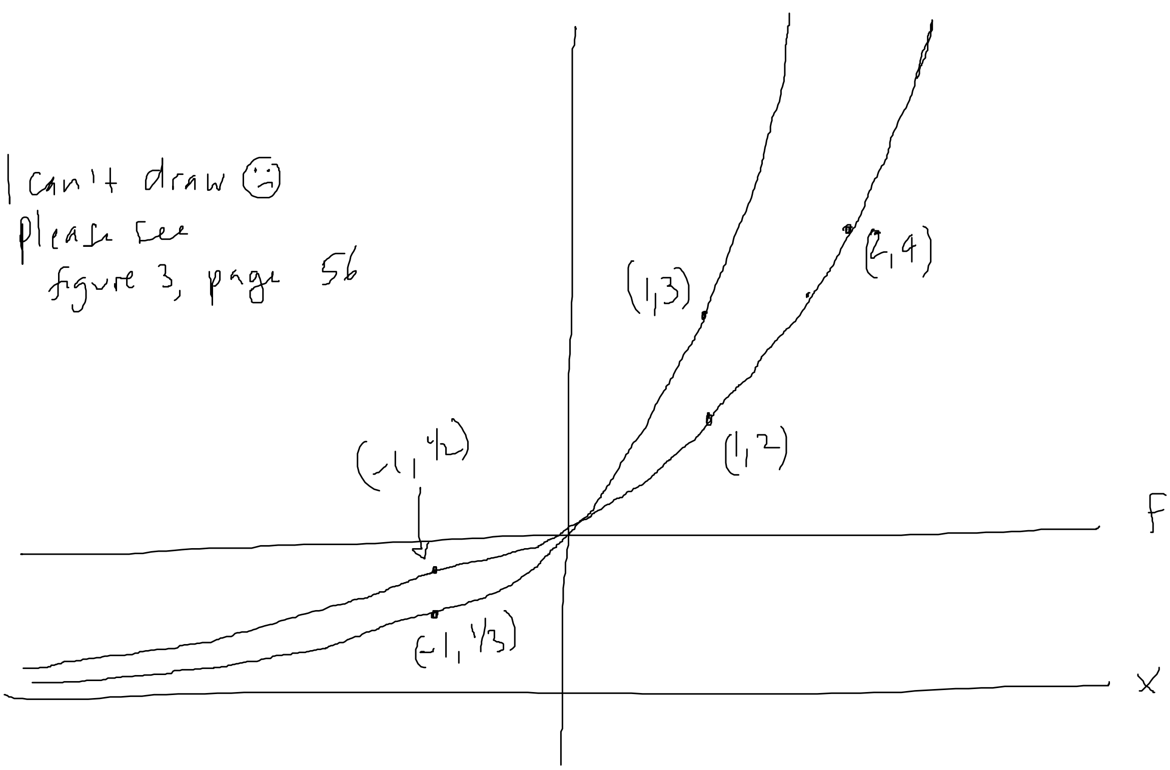
	$f(x)$	$g(x)$	$h(x)$
$x=0$	1	1	1
$x=1$	1	2	3
$x=2$	1	4	9
$x=3$	1	8	27
$x=-1$	1	1/2	1/3
$x=-2$	1	1/4	1/9

So f is constant.

Both g and h get bigger and bigger as x gets bigger, but h grows more quickly.

Both g and h get smaller and smaller as x gets more and more negative, but h goes to zero more quickly.

I can't draw please see figure 3, page 56

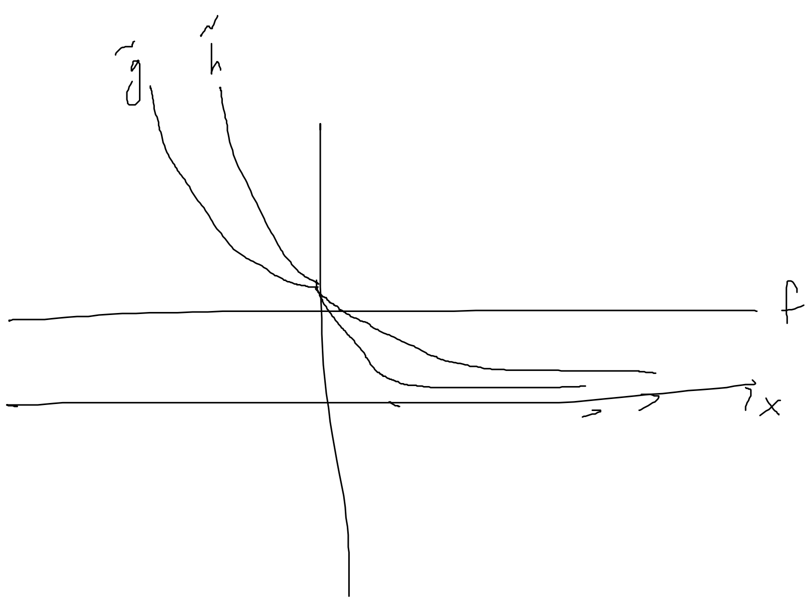


$$f(x) = 1^x$$
$$\tilde{g}(x) = (1/2)^x$$
$$\tilde{h}(x) = (1/3)^x$$

	$f(x)$	$\tilde{g}(x)$	$\tilde{h}(x)$
$x = 0$	1	1	1
$x = 1$	1	1/2	1/3
$x = 2$	1	1/4	1/9
$x = 3$	1	1/16	1/27
$x = -1$	1	2	3
$x = -2$	1	4	9

In this case we see that both $\tilde{g}$ and $\tilde{h}$ get smaller as x gets bigger, with $\tilde{h}$ getting small faster than $\tilde{g}$.

And as x gets more + more negative, $\tilde{g}$ and $\tilde{h}$ get bigger + bigger w/ $\tilde{h}$ growing faster



again, please see figure 3 on page 56!

Q: if I reflect the graph of $f(x) = 2^x$ about the y-axis, do I get the graph of $\tilde{f}(x) = (1/2)^x$?

A: yes! Reflecting about the y-axis means replacing x with -x. So the graphs will be the same if

$$f(-x) = \tilde{f}(x).$$

$$f(-x) = 2^{-x} = (2^{-1})^x = (1/2)^x = \tilde{f}(x). \checkmark$$

You should check that you know how to plot things like $-2 \cdot 3^x$ and $2^x - 5$ and 5^{x-2} and $(1/2)^{x+1}$ and the like.

laws of exponents:

$$a^{x+y} = a^x a^y$$

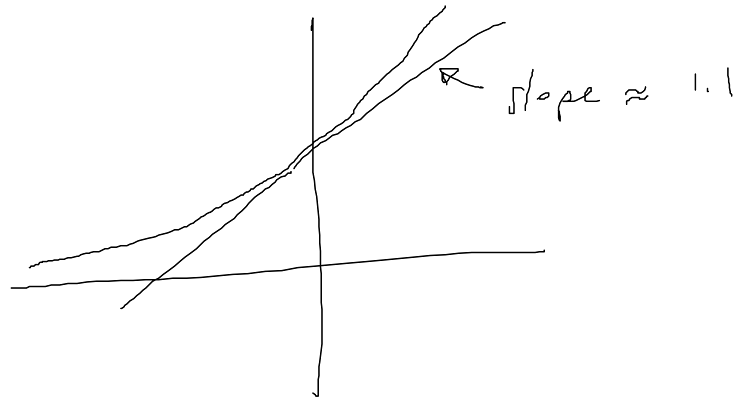
$$a^{x-y} = \frac{a^x}{a^y}$$

$$(a^x)^y = a^{xy}$$

$$(ab)^x = a^x b^x$$

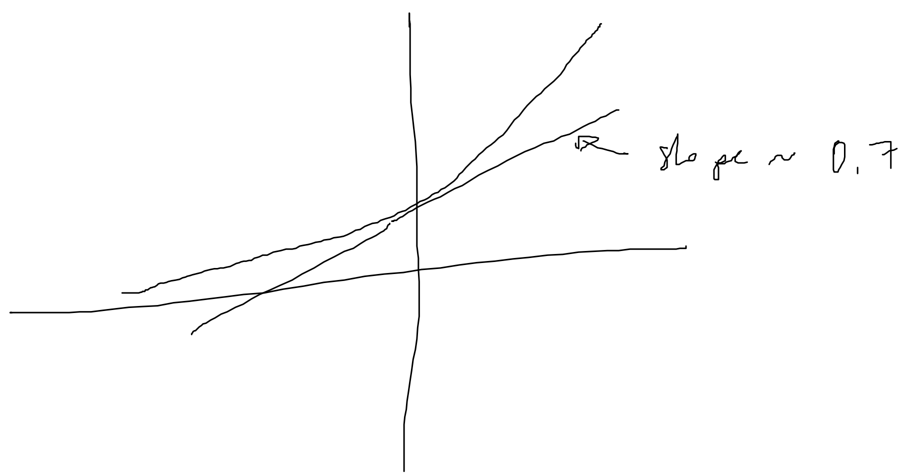
If you look at the graph of

$f(x) = 3^x$

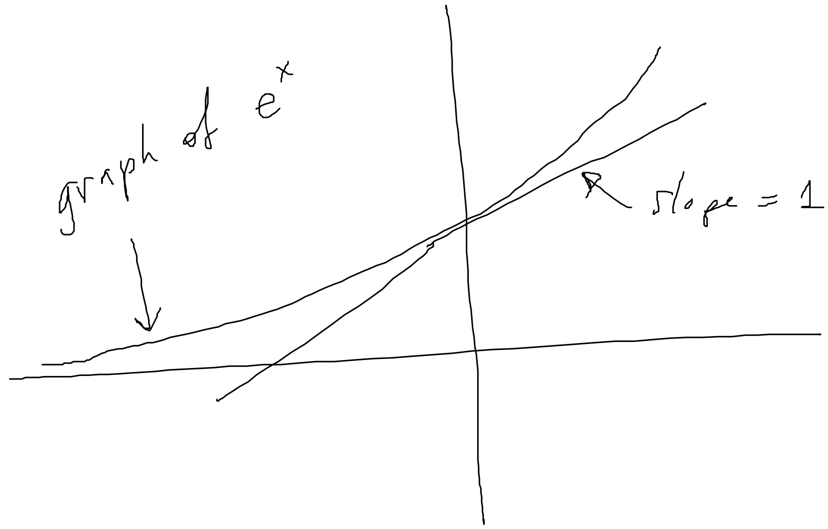


and look at the tangent line to the graph through $(0,1)$ then its slope is approximately 1.1

For the graph of $g(x) = 2^x$, the slope of the tangent line through $(0,1)$ is approximately 0.7



There is a number between 2 and 3 so that the slope of the tangent line through $(0,1)$ is <u>exactly</u> 1. We call this number "e"



You can approximate e on your calculator

$e \cong 2.71828$

but this is an approximation. e will be just as important as π in your lives.

ex #13 Start with the graph of e^x. Write the equation of the graph that results from...

a) shifting 2 units downwards.

ans: $f(x) = e^x - 2$

b) shifting 2 units to the right

ans: $f(x) = e^{x-2}$

c) reflecting about the x-axis

ans: $f(x) = -e^x$

d) reflecting about the y-axis

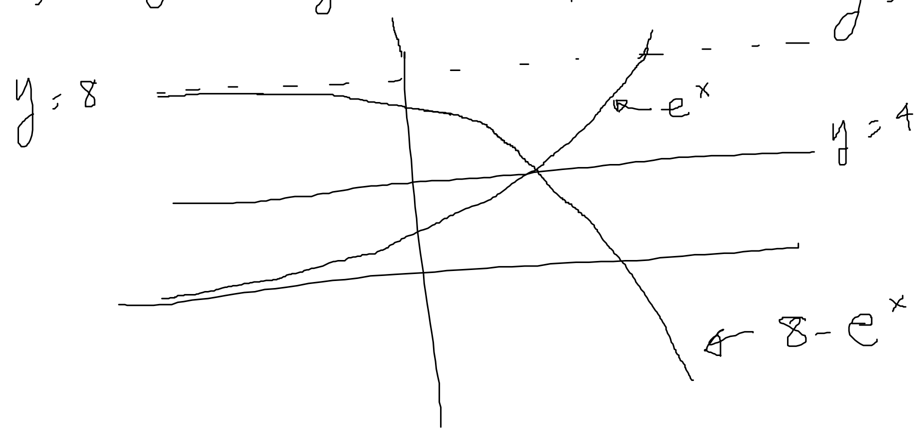
<u>ans:</u> $f(x) = e^{-x}$ or $(1/e)^x$

e) reflecting about the x axis and then about the y-axis

<u>ans:</u> $f(x) = -e^{-x}$

<u>ex 14:</u> Starting with the graph of e^x, find the equation of the graph that results from

a) reflecting about the line $y = 4$

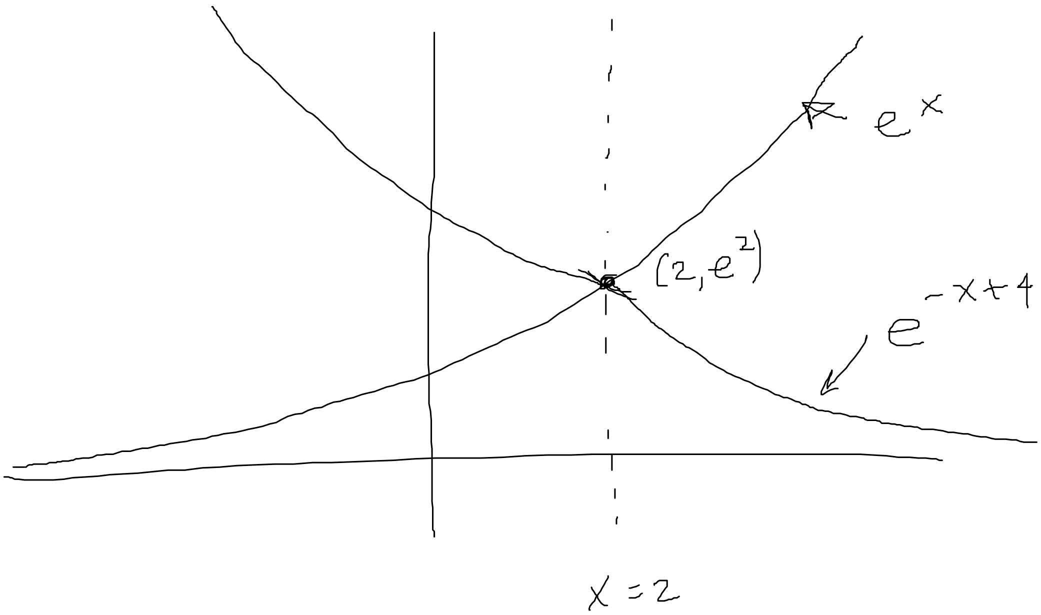


Why? take graph, lower by 4, reflect about x axis, then raise by 4.

i.e. $e^x \to e^x - 4 \to -e^x + 4 \to -e^x + 8 = 8 - e^x$

lower by 4 — reflect about x-axis — raise by 4.

b) reflecting about the line $\underline{x = 2}$



take graph, move left by 2, reflect about y axis, move right by 2.

$$e^x \to e^{x+2} \to e^{-x+2} \to e^{-(x-2)+2} = e^{-x+4}$$

why do I believe this? well,

$$f(x) = e^x \quad g(x) = e^{-x+4}$$

$g(x)$ goes to 0 as x gets bigger and bigger ✓

$\underline{\text{and}}$ $f(2) = e^2 \quad g(2) = e^2$

they cross at $(2, e^2)$ as desired

#19 if $f(x) = 5^x$ show that

$$\frac{f(x+h) - f(x)}{h} = 5^x\left(\frac{5^h - 1}{h}\right)$$

$$\frac{f(x+h) - f(x)}{h} = \frac{5^{x+h} - 5^x}{h} = \frac{5^x 5^h - 5^x}{h}$$

$$= \frac{5^x(5^h - 1)}{h} \checkmark$$

#21 Suppose the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch. Show that at a distance 2 feet to the right of the origin, the height of the graph of f is 48 feet but the height of the graph of g is about 265 miles.

2 feet $= 2 \times 12$ inches $= 24$.

$f(24) = (24)^2 = 576$ inches $= 576$ inches $\times \frac{1 \text{ foot}}{12 \text{ inches}} = 48$ feet ✓

$g(24) = 2^{24} = 16777216$ inches

$= 16777216 \text{ in} \times \frac{\text{ft}}{12 \text{ in}} \times \frac{\text{mile}}{5280 \text{ feet}} = \frac{16777216}{63360} \cong 264.8$ mile

#20 Suppose you're offered a job that lasts one month. Which of the following methods of payment do you prefer?

(a) one million \$ at the end of the month

(b) one cent on day 1, 2¢ on day 2, 4¢ on day 3, 2^{n-1} ¢ on day n.

$$1+2+4+8+\dots+2^k$$

$$= 1+2^1+2^2+2^3+\dots+2^k$$

$$= \frac{1-2^{k+1}}{1-2}$$

If there are 30 days in the month, then total pay $= 1+2+2^2+\dots+2^{29} = \frac{1-2^{30}}{1-2} \cong 1.07\times 10^9$ ¢

$= 1.07\times 10^7$ \$

$= 10{,}700{,}000$ \$

you're better off with plan (b).