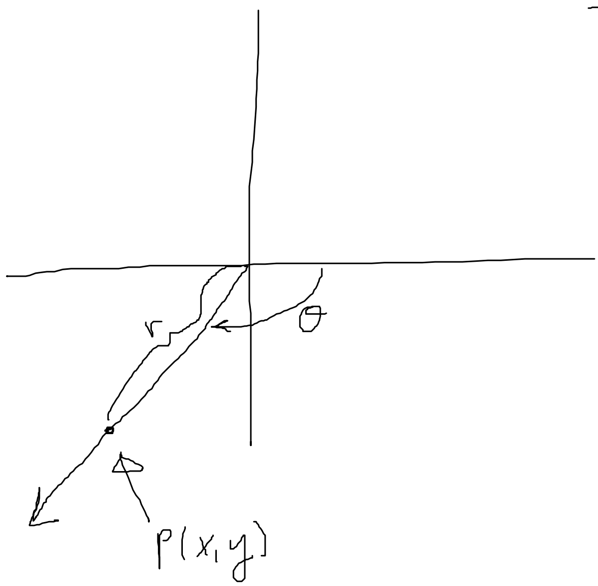
Mat 135 Monday Sept 13, 2004

We define the trig functions for $0 < \theta < \pi/2$ in terms of the lengths of sides of right triangles. What about for other angles?

Given an angle θ in standard position, choose a point $P(x,y)$ on the terminal side of the angle. Let r be the distance from $P(x,y)$ to $P(0,0)$.



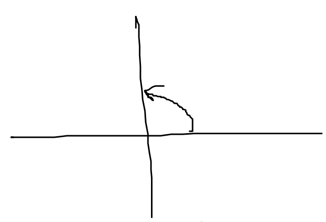
Then

$$\sin(\theta) = y/r$$
$$\cos(\theta) = x/r$$
$$\tan(\theta) = y/x$$

and

$$\csc(\theta) = r/y$$
$$\sec(\theta) = r/x$$
$$\cot(\theta) = x/y.$$

for example $\theta = \pi/2$

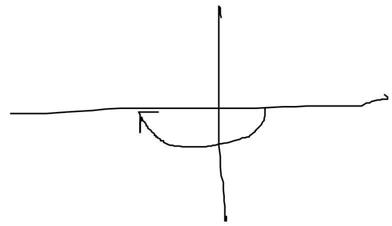


$P(0,5)$ is on the terminal side. $r = 5$

$\sin(\pi/2) = 5/5 = 1$ $\quad \cos(\pi/2) = 0/5 = 0$ $\quad \tan(\pi/2)$ is not defined

$\csc(\pi/2) = 1$ $\quad \sec(\pi/2)$ not defined $\quad \cot(\pi/2) = 0$

$\theta = -\pi$

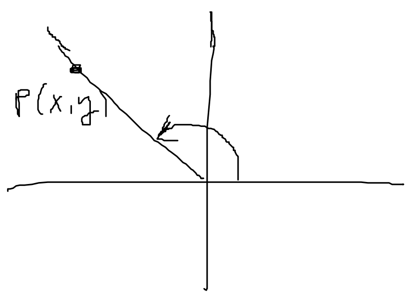


$P(-21, 0)$ is on the terminal side. $r = 21$ in this case.

$\Rightarrow \sin(-\pi) = 0/21 = 0 \quad \cos(-\pi) = \frac{-21}{21} = -1 \quad \tan(-\pi) = 0/-1 = 0$

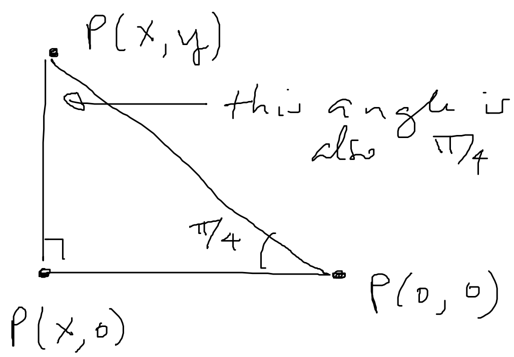
$\csc(-\pi)$ not defined, $\sec(-\pi) = -1$ $\cot(-\pi)$ not defined.

$\theta = \frac{3\pi}{4}$



how can we find a point $P(x,y)$? Use geometry.

$\frac{3\pi}{4} = \pi - \frac{\pi}{4} \quad \Rightarrow$



$\Rightarrow$ adj $= |x|$
opp $= y$
hyp $= \sqrt{x^2+y^2}$

but the triangle has opp = adj

$\Rightarrow x = -y$

$\Rightarrow P(-1,1)$ is on the terminal side

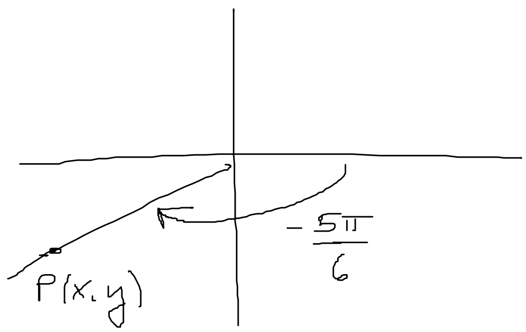
So $P(-1,1)$ is on terminal side. $r = \sqrt{2}$ in this case.

$$\Rightarrow \quad \sin(3\pi/4) = +1/\sqrt{2} \qquad \csc\left(\frac{3\pi}{4}\right) = \sqrt{2}$$

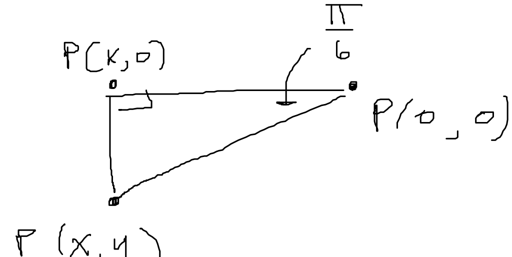
$$\cos(3\pi/4) = -1/\sqrt{2} \qquad \sec\left(\frac{3\pi}{4}\right) = -\sqrt{2}$$

$$\tan(3\pi/4) = 1/-1 = -1 \qquad \cot(3\pi/4) = -1$$

$\theta = -\dfrac{5\pi}{6}$



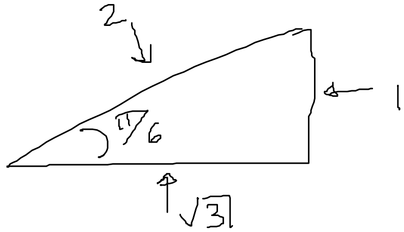
since $\pi - \frac{5\pi}{6} = \frac{\pi}{6}$ we have



$\Rightarrow$ adj $= -x$
opp $= -y$
hyp $= \sqrt{x^2+y^2}$

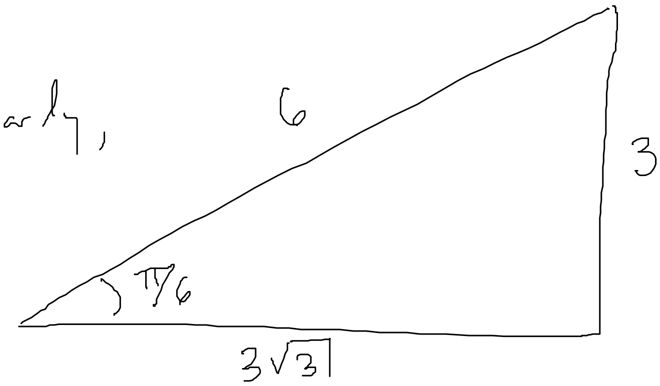
how can we find x & y that work?

We know



So $x = -\sqrt{3}$
$y = -1$
works. In this case, $r = 2$

Similarly,



So $x = -3\sqrt{3}$
$y = -3$
works. In this case, $r = 6$

Whichever $P(x, y)$ you choose,

$\sin\left(-\frac{5\pi}{6}\right) = -1/2$ $\qquad$ $\csc\left(-\frac{5\pi}{6}\right) = -2$

$\cos\left(-\frac{5\pi}{6}\right) = -\sqrt{3}/2$ $\qquad$ $\sec\left(-\frac{5\pi}{6}\right) = -2/\sqrt{3}$

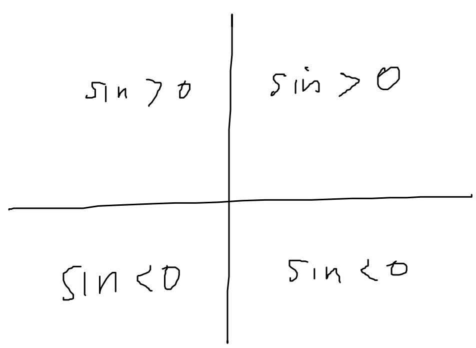
$\tan\left(-\frac{5\pi}{6}\right) = 1/\sqrt{3}$ $\qquad$ $\cot\left(-\frac{5\pi}{6}\right) = \sqrt{3}$

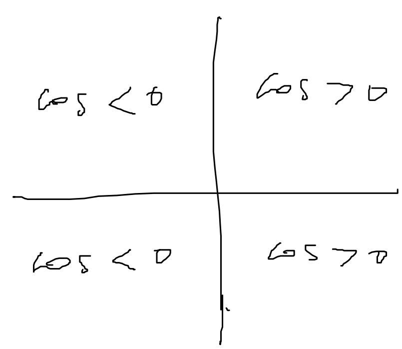
Fact: Given any θ of the form

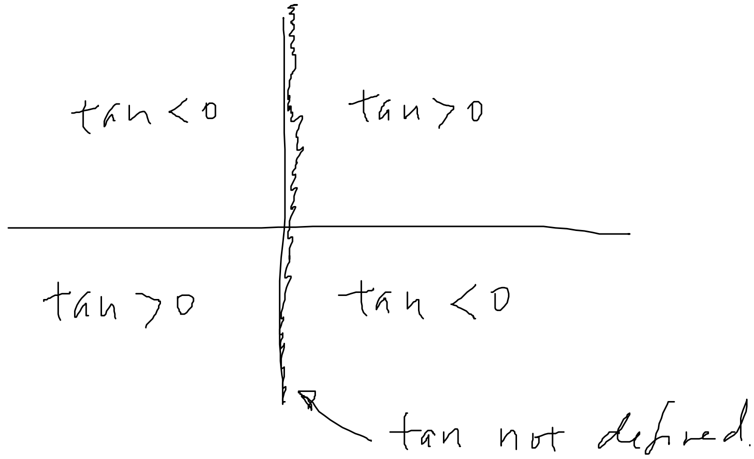
$\frac{k\pi}{2}$ or $\frac{k\pi}{3}$ or $\frac{k\pi}{6}$ or $k\pi$

where k is some integer $(k = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots)$ you should be able to find all 6 trig functions (if defined) by working carefully and drawing triangles.

Fact:







Q: If I'm good at memorizing things, can I find $\sin(-\pi/3)$ and $\cos\left(\frac{5\pi}{6}\right)$ and stuff in a faster way?

A: Yes!

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

So...

$$\sin(5\pi/6) = \sin(\pi/2 + \pi/3)$$

$$= \sin(\pi/2)\cos(\pi/3) + \cos(\pi/2)\sin(\pi/3)$$

$$= 1 \cdot \cos(\pi/3) + 0 \cdot \sin(\pi/3)$$

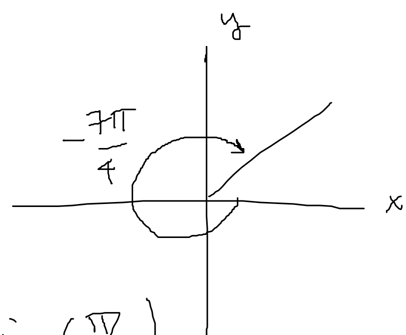
$$= \cos(\pi/3) = 1/2$$

$$\cos\left(\frac{3\pi}{4}\right) = \cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$$

$$= \cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)$$

$$= 0 \cdot \cos\left(\frac{\pi}{4}\right) - 1 \cdot \sin(\pi/4)$$

$$= -1/\sqrt{2}$$

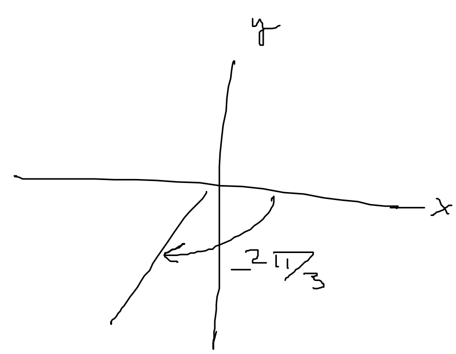


$$\sin\left(-\frac{7\pi}{4}\right) = \sin\left(-2\pi + \frac{\pi}{4}\right)$$

$$= \sin(-2\pi)\cos\left(\frac{\pi}{4}\right) + \cos(-2\pi)\sin(\pi/4)$$

$$= 0 \cdot \cos(\pi/4) + 1 \cdot \sin(\pi/4)$$

$$= 1/\sqrt{2}$$



$$\tan\left(-\frac{2\pi}{3}\right) = \tan(-\pi + \pi/3)$$

$$= \frac{\sin(-\pi + \pi/3)}{\cos(-\pi + \pi/3)}$$

$$= \frac{\sin(-\pi)\cos(\pi/3) + \cos(-\pi)\sin(\pi/3)}{\cos(-\pi)\cos(\pi/3) - \sin(-\pi)\sin(\pi/3)}$$

$$= \frac{0 \cdot \cos(\pi/3) - 1 \cdot \sin(\pi/3)}{-1 \cdot \cos(\pi/3) - 0 \cdot \sin(\pi/3)} = \frac{-\sqrt{3}/2}{-1/2} = \sqrt{3}$$

Q: If I've figured out $\sin(\theta)$, is there a fast way for me to find $\cos(\theta)$? And vice versa?

A: Yes!

You know $\sin(\theta) = \frac{y}{r}$ $\cos(\theta) = \frac{x}{r}$ where $P(x,y)$ is some point on the terminal side of θ in standard position.

$$\Rightarrow \left(\sin(\theta)\right)^2 + \left(\cos(\theta)\right)^2 = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2$$

$$= \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2+y^2}{r^2} = \frac{r^2}{r^2} = 1.$$

$$\Rightarrow \left(\sin(\theta)\right)^2 + \left(\cos(\theta)\right)^2 = 1.$$

Note: the book writes this as

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

(the exponents are in a different place)

So...

$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$ (from before)

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\cos\left(\frac{5\pi}{6}\right)\right)^2 = 1$$

$$\Rightarrow \left(\cos\left(\frac{5\pi}{6}\right)\right)^2 = \cancel{4}\ 1 - \frac{1}{4} = 3/4 \Rightarrow \cos\left(\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{2} \text{ or } -\frac{\sqrt{3}}{2}$$

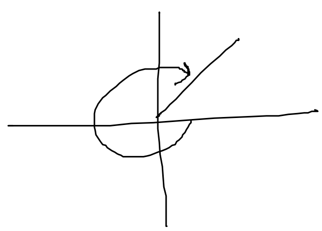
Which?? $\frac{5\pi}{6}$ is  so $\cos(5\pi/6) < 0 \Rightarrow \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

Similarly, $\sin\left(-\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}}$ from before

$$\Rightarrow \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\cos\left(-\frac{7\pi}{4}\right)\right)^2 = 1$$

$$\Rightarrow \left(\cos\left(-\frac{7\pi}{4}\right)\right)^2 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \cos\left(-\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}} \text{ or } \frac{-1}{\sqrt{2}}.$$ Which one?



$\cos\left(-\frac{7\pi}{4}\right) > 0$

so $\cos\left(-\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}}$

You must learn 3 identities:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

Also, you need to know that

$$\sin(-\theta) = -\sin(\theta) \text{ and } \cos(-\theta) = \cos(\theta)$$

which follows from drawing a graph.

all the others on pp A28-A30
follow from these

for example,

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$

why?

$$\begin{aligned}\sin(x-y) &= \sin(x+(-y)) \\ &= \sin(x)\cos(-y) + \cos(x)\sin(-y) \\ &= \sin(x)\cos(y) - \cos(x)\sin(y)\end{aligned}$$

since cos is an even function and sin is an odd function

For example,

$$\sin(2x) = 2\sin(x)\cos(x)$$

why?

$$\begin{aligned}\sin(2x) &= \sin(x+x) \\ &= \sin(x)\cos(x) + \cos(x)\sin(x) \\ &= 2\sin(x)\cos(x)\end{aligned}$$

For example,

$$\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

why?

$$\begin{aligned}\cos(2x) &= \cos(x+x) \\ &= \cos(x)\cos(x) - \sin(x)\sin(x) \\ &= \cos^2 x - \sin^2 x \\ &= \cos^2 x - [1-\cos^2 x] = 2\cos^2 x - 1\end{aligned}$$

or

$$= [1-\sin^2 x] - \sin^2 x = 1 - 2\sin^2 x$$