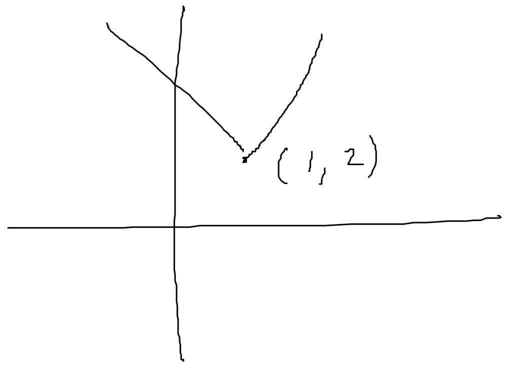
Mat 135 Oct 8, 2004

Consider $f(x) = \begin{cases} x^2+1 & x \geq 1 \\ -x+3 & x < 1 \end{cases}$



then want to know if f is differentiable at $a=1$.

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \quad \text{exists?}$$

$$\lim_{h \to 0+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0+} \frac{[(1+h)^2+1] - [1^2+1]}{h}$$

$$= \lim_{h \to 0+} \frac{[1+2h+h^2+1] - [2]}{h}$$

$$= \lim_{h \to 0+} \frac{2h+h^2}{h} = \lim_{h \to 0} 2+h = 2$$

the limit of the secant line slopes from the right is >0, as expected from the graph

$$\lim_{h \to 0-} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0-} \frac{[-(1+h)+3]-[1^2+1]}{h}$$

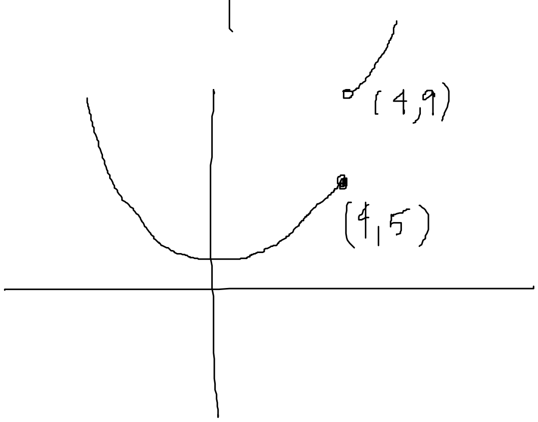
$$= \lim_{h \to 0^-} \frac{[-1-h+3]-[2]}{h} = \lim_{h \to 0-} \frac{-h}{h} = 1$$

the limit of the secant line slopes from the left is < 0 as expected

Since $\lim\limits_{h \to 0+} \neq \lim\limits_{h \to 0-}$ $\lim\limits_{h \to 0} \frac{f(1+h)-f(1)}{h}$

does not exist and f is not differentiable at $a=1$.

Ex: $f(x) = \begin{cases} x^3 - 55 & x > 4 \\ x^2 - 11 & x \leq 4 \end{cases}$



clearly, f is not differentiable at $a=4$. What goes wrong?

$$\lim_{h \to 0-} \frac{f(4+h)-f(4)}{h} = \lim_{h \to 0-} \frac{[(4+h)^2-11]-5}{h}$$

$$= \lim_{h \to 0^-} \frac{[16+8h+h^2-11]-5}{h}$$

$$= \lim_{h \to 0^-} \frac{8h+h^2}{h} = \lim_{h\to 0^-} 8+h = 8$$

the secant line slopes from the left have a finite value since there's no jump in the value at $a=4$.

$$\lim_{h\to 0^+} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0^+} \frac{[(4+h)^3-55]-5}{h}$$

$$= \lim_{h\to 0^+} \frac{[64+48h+12h^2+h^3-55]-5}{h}$$

$$= \lim_{h\to 0^+} \frac{48h+12h^2+h^3+4}{h}$$

$$= \lim_{h\to 0^+} 48+12h+h^2+\frac{4}{h}$$

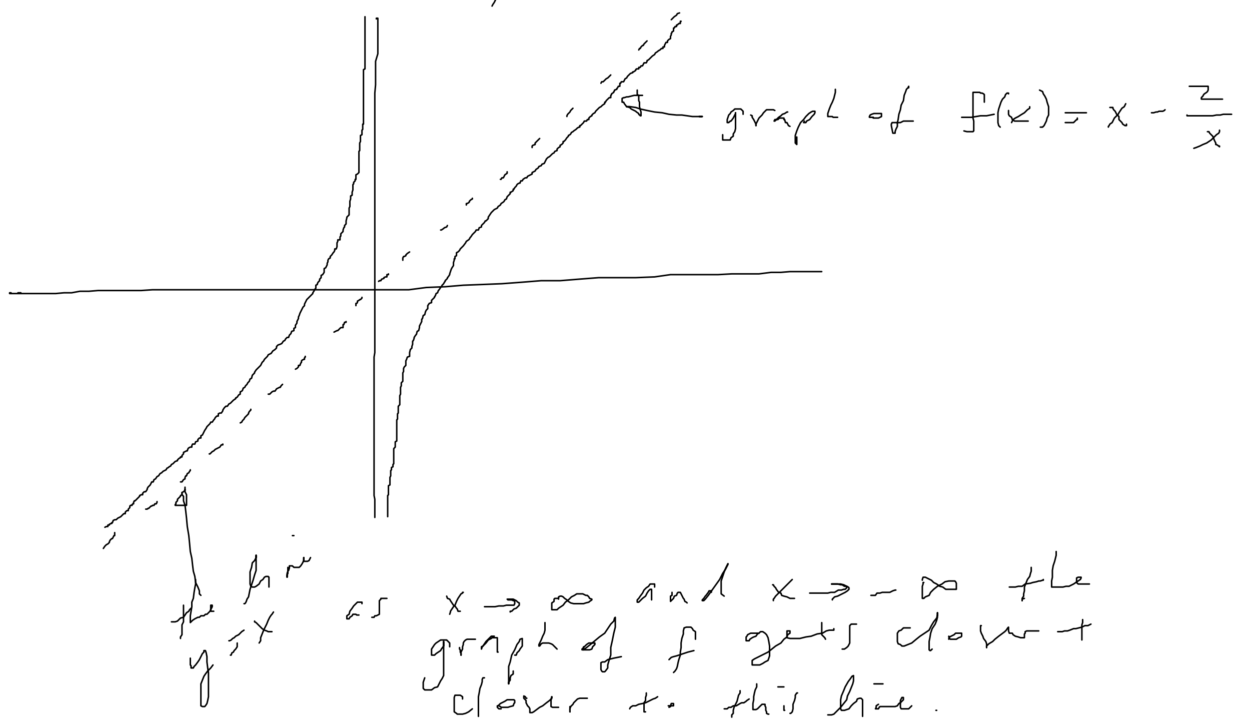
this limit doesn't exist since as h decreases to zero, $\frac{4}{h}$ increases towards infinity.

$$\Rightarrow \lim_{h\to 0^+} \frac{f(4+h)-f(4)}{h} = \infty.$$

Again, this outcome is what we expected from the graph of f.

ex: $f(x) = x - \frac{2}{x}$

compute its derivative and decide if the derivative's what you expected



as $x \to \infty$ and $x \to -\infty$ the graph of f gets closer + closer to this line.

From the graph of f, what do we expect for f'?

1. as $x \to \infty$, $f' \to 1$ because the graph of f is getting closer + closer to the graph of $g(x) = x$. Which has derivative 1.
2. as $x \to -\infty$, $f' \to 1$ for the same reason
3. $f' > 0$ at all points except $x = 0$. $f'(0)$ not defined
4. as $x \to 0-$, $f'(x) \to +\infty$
5. as $x \to 0+$, $f'(x) \to -\infty$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{\left[(a+h) - \frac{2}{a+h}\right] - \left[a - \frac{2}{a}\right]}{h}$$

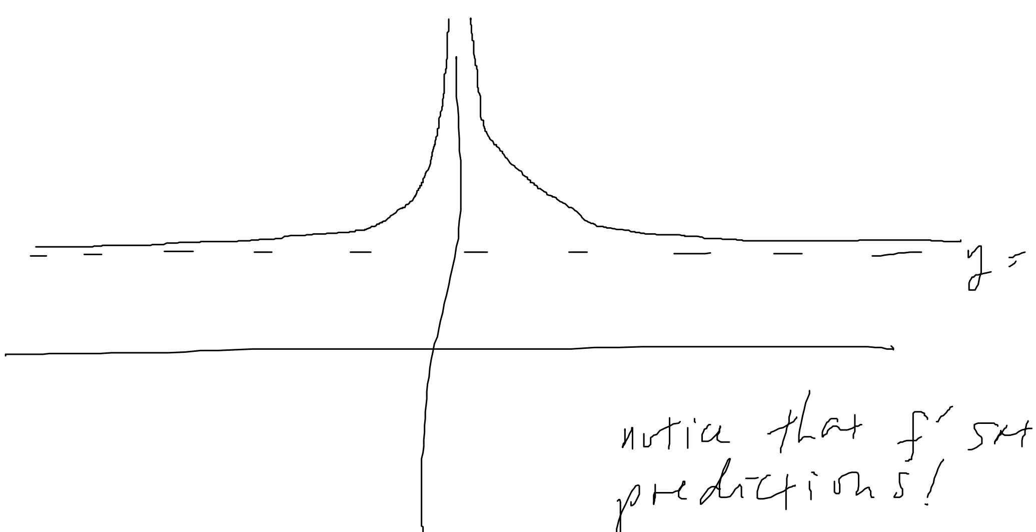
$$= \lim_{h\to 0} \frac{[(a+h) - a] + [2/a - 2/a+h]}{h}$$

$$= \lim_{h\to 0} \frac{h + \frac{2(a+h) - 2a}{a(a+h)}}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} + \frac{2a + 2h - 2a}{ha(a+h)}$$

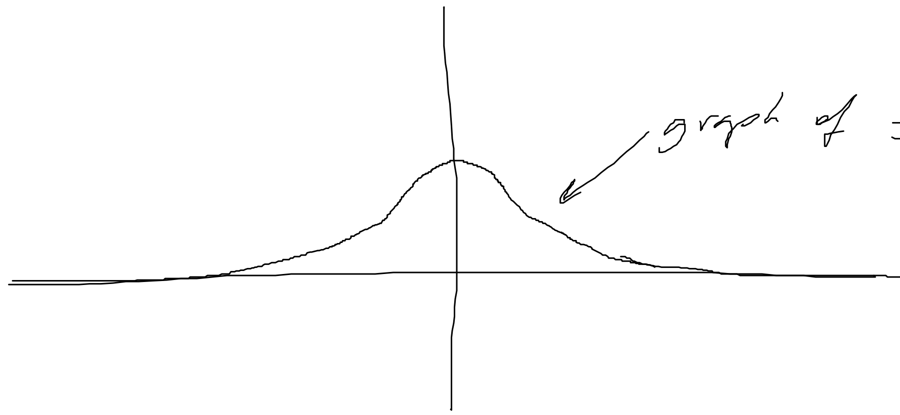
$$= \lim_{h\to 0} 1 + \frac{2}{a(a+h)} = 1 + \frac{2}{a^2}$$

And so $f'(a) = 1 + \frac{2}{a^2}$



notice that f' satisfies all predictions!

ex: $f(t) = \frac{6}{1+t^2}$



predictions: ① as $a \to \infty$, $f'(a) \to 0$ since the graph of f is getting closer + closer to a horizontal line ($y=0$, specifically) and a constant function has zero derivative ($\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0} \frac{c-c}{h} = 0.$)

② as $a \to -\infty$, $f'(a) \to 0$ as well.

③ $f'(a)$ defined for all a.

④ $f' > 0$ for $a < 0$

⑤ $f' < 0$ for $a > 0$

⑥ $f'(0) = 0$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{6}{1+(a+h)^2} - \frac{6}{1+a^2}}{h}$$

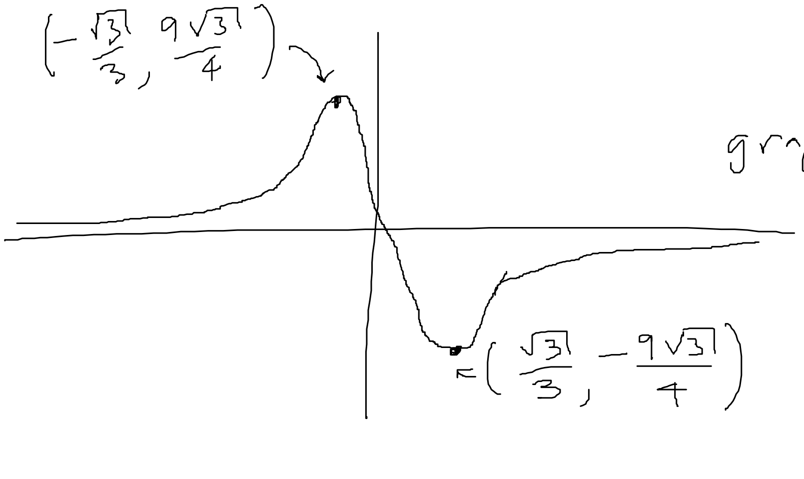
$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{6(1+a^2) - 6(1+(a+h)^2)}{[1+(a+h)^2][1+a^2]}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{6+6a^2 - 6(1+a^2+2ah+h^2)}{[1+(a+h)^2][1+a^2]}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{6+6a^2-6-6a^2-12ah-6h^2}{[1+(a+h)^2][1+a^2]}\right]$$

$$= \lim_{h \to 0} \frac{-12a-6h}{[1+(a+h)^2][1+a^2]} = -\frac{12a}{(1+a^2)^2}$$

So $f'(a) = -\dfrac{12a}{(1+a^2)^2}$ } It satisfies all our expectations! How thrilling!!



graph of f'