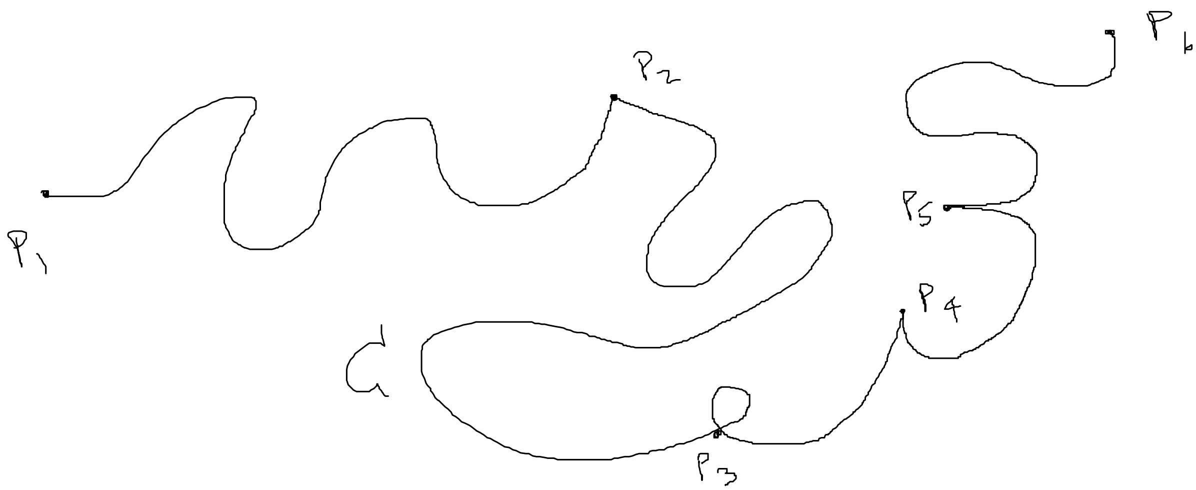
Mat 135 Oct 6, 2004

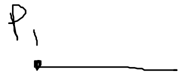
If P is a point on a curve C, then there is a tangent line to C at P if: the closer and closer that you "zoom in" to the point P, the more and more the curve looks like a straight line running through P.

(Just as the Earth is not flat when seen from space, if you "zoom in" on the earth it certainly looks flat; when you're standing on the surface of the earth, it's as if you're just a point on a plane. It's the same thing with curves except in this calc. course we live in 2 dimensions rather than 3 dimensions. So we think about lines and curves, rather than planes and surfaces.)

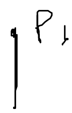
For the following curve, what are the points where there is no tangent line?



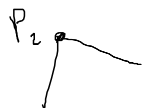
no tangent lines at $P_1, P_2, P_3, P_4, P_5, P_6$. Why? If you look really close at these points, C looks like

P_1: 

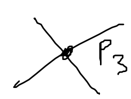
the line dead-ends, it doesn't run through P_1

P_6: 

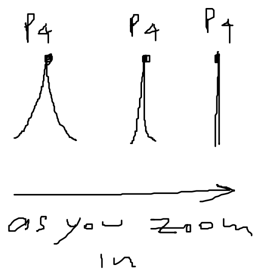
the line dead-ends, it doesn't run through P_6

P_2: 

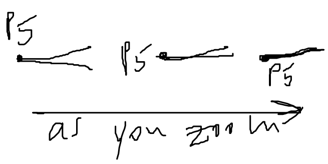
it looks like a corner --- the curve doesn't look like a line running through P_2

P_3: 

it looks like an X --- the curve doesn't look like a line running through P_3

P_4: 

the curve approaches and then doubles back on itself --- the curve doesn't look like a line running through P_4

P_5 

see the explanation for P_4.

§2.8 Derivatives

definition: the derivative of a function f at a, denoted by $f'(a)$ is

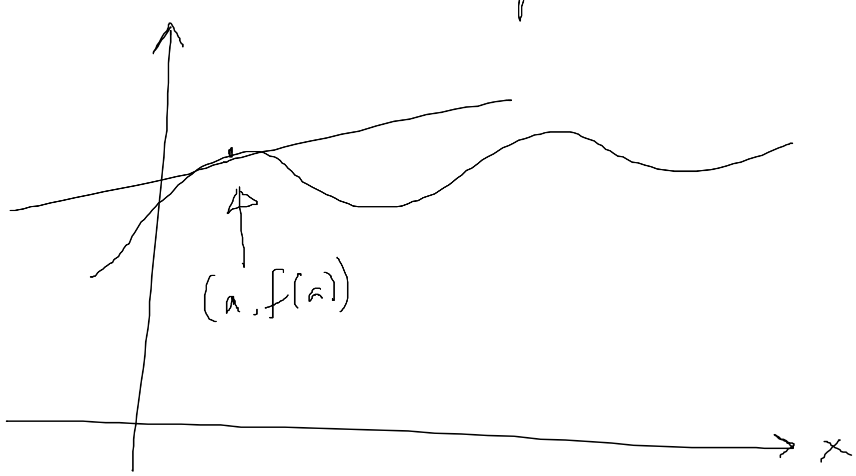
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Alternately, this can be written as

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

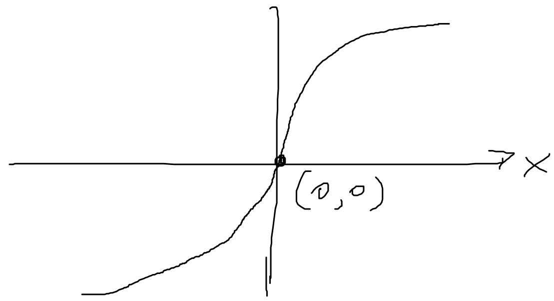
From last time, if G is the graph of a function f and $(a, f(a))$ is a point on G then the slope of the tangent line to G at $(a, f(a))$ is precisely $f'(a)$



tangent line: $\boxed{y = f'(a)(x-a) + f(a)}$

From last time;

$f(x) = \sqrt[3]{x}$



there is a tangent line at $(0,0)$: $\boxed{x=0}$ is the tangent line.

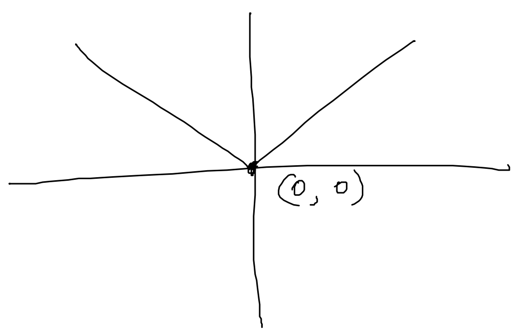
But $f'(0)$ does not exist. When we calculated

$$\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \infty$$

Note: If you read the book very carefully then $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \infty$ is viewed as "the limit does not exist". And so $f'(0)$ does not exist. For more, see page 99, the paragraph beneath figure 11.

From last time,

$f(x) = |x|$



there is no tangent line at $(0,0)$

and $f'(0)$ does not exist. When we calculated

$$\lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$$

we got $\lim_{h \to 0^+} = 1$

and $\lim_{h \to 0^-} = -1$

ex: find $f'(a)$ for $f(t) = t^4 - 5t$

$$f'(a) = \lim_{h\to 0} \frac{[(a+h)^4 - 5(a+h)] - [a^4 - 5a]}{h}$$

$$= \lim_{h\to 0} \frac{[a^4 + 4a^3h + 6a^2h^2 + 4ah^3 + h^4 - 5a - 5h] - [a^4 - 5a]}{h}$$

$$= \lim_{h\to 0} \frac{4a^3h + 6a^2h^2 + 4ah^3 + h^4 - 5h}{h}$$

$$= \lim_{h\to 0} 4a^3 + 6a^2h + 4ah^2 + h^3 - 5$$

$$= 4a^3 - 5$$

ex: find $f'(a)$ for $f(t) = \frac{2t+1}{t+3}$

$$f'(a) = \lim_{h\to 0} \frac{\frac{2(a+h)+1}{(a+h)+3} - \frac{2a+1}{a+3}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{2a+2h+1}{a+h+3} - \frac{2a+1}{a+3}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(2a+2h+1)(a+3) - (2a+1)(a+h+3)}{(a+h+3)(a+3)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{5h}{(a+h+3)(a+3)}$$

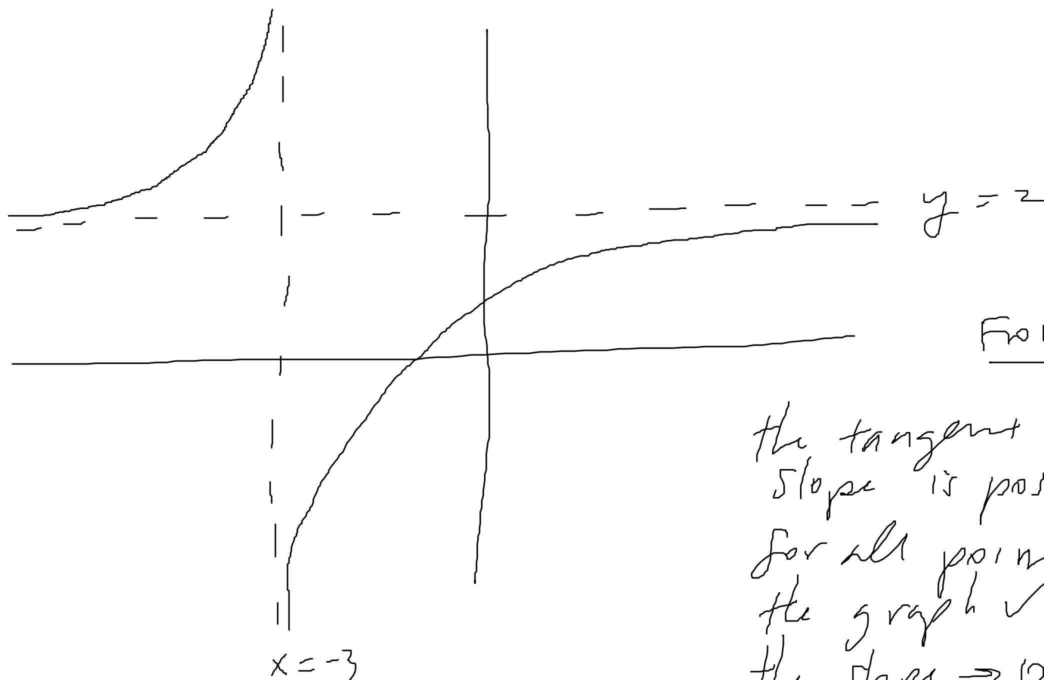
$$= \lim_{h\to 0} \frac{5}{(a+h+3)(a+3)}$$

$$= \frac{5}{(a+3)^2}$$

$$f'(a) = \frac{5}{(a+3)^2}$$

does it make sense that $f'(a) > 0$ for all $a \neq -3$?

$$f(t) = \frac{2t+1}{t+3} = 2 - \frac{5}{t+3}$$



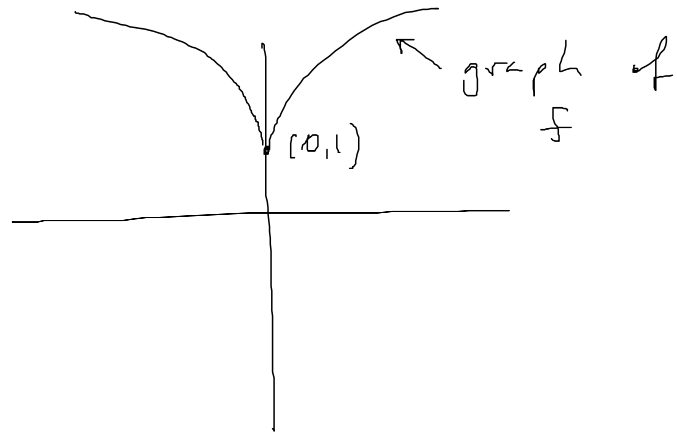
<u>From the graph:</u>

the tangent line slope is positive for all points on the graph ✓

the slope $\to 0$ as $a \to \infty$ and $a \to -\infty$ ✓

the slope $\to \infty$ as $a \to (-3)^+$ and $a \to (-3)^-$ ✓

$$f(x) = \sqrt{|x|} + 1$$



have no tangent line at (0, 1)

have tangent line at (4, 3) and at (−4, 3)

to find their slopes, we need to compute $f'(a)$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\left[\sqrt{|a+h|} + 1\right] - \left[\sqrt{|a|} + 1\right]}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{|a+h|} - \sqrt{|a|}}{h}$$

we know $|a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } a < 0 \end{cases}$

$$f'(4) = \lim_{h \to 0} \frac{\sqrt{|4+h|} - \sqrt{|4|}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{4+h} - \sqrt{4}}{h}\right)\left(\frac{\sqrt{4+h} + \sqrt{4}}{\sqrt{4+h} + \sqrt{4}}\right)$$

Q: certainly $|4| = 4$. But why $|4+h| = 4+h$?

A: if h is getting really small then even if $h < 0$, you'll still have $4 + h > 0$

$$f'(4) = \lim_{h\to 0} \frac{(4+h)-4}{h(\sqrt{4+h}+\sqrt{4})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+\sqrt{4}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

note: $f'(4) > 0$ as expected from graph.

$$f'(-4) = \lim_{h\to 0} \frac{\sqrt{|-4+h|}-\sqrt{|-4|}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{4-h}-\sqrt{4}}{h}$$

Q: huh? A: since $-4<0$, if h is getting really small then $-4+h<0$ so $|-4+h| = -(-4+h) = 4-h$

$$= \lim_{h\to 0} \left(\frac{\sqrt{4-h}-\sqrt{4}}{h}\right)\left(\frac{\sqrt{4-h}+\sqrt{4}}{\sqrt{4-h}+\sqrt{4}}\right)$$

$$= \lim_{h\to 0} \frac{(4-h)-4}{h(\sqrt{4-h}+\sqrt{4})} = \lim_{h\to 0} \frac{-1}{\sqrt{4-h}+\sqrt{4}} = \frac{-1}{4}$$

note: $f'(-4) < 0$ as expected from graph.

$$f'(0) = \lim_{h\to 0} \frac{\sqrt{|0+h|}-\sqrt{|0|}}{h} = \lim_{h\to 0} \frac{\sqrt{|h|}}{h}$$

$$\lim_{h\to 0+} \frac{\sqrt{|h|}}{h} = \lim_{h\to 0+} \frac{\sqrt{h}}{h} = \lim_{h\to 0+} \frac{1}{\sqrt{h}} = \infty$$ as expected from graph. ✓

$$\lim_{h\to 0-} \frac{\sqrt{|h|}}{h} = \lim_{h\to 0-} \frac{\sqrt{-h}}{-(-h)} = \lim_{h\to 0-} \frac{-1}{\sqrt{-h}} = -\infty$$ as expected from graph. ✓